Introduction and basic aspects of wavelets theory

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A Characterization of the Higher Dimensional Groups Associated With Continuous Wavelets.

R.S. Laugesen, N. Weaver, G.L. Weiss and E.N. Wilson [1]

Abstract

A subgroup D of $GL(n, \mathbb{R})$ is said to be *admissible* if the semidirect product of D and $\mathbb{R}^n$, considered as a subgroup of the affine group on $\mathbb{R}^n$, admits wavelets $\psi \in L^2(\mathbb{R}^n)$ satisfying a generalization of the Calderón reproducing formula. This paper provides a nearly complete characterization of the admissible subgroups D. More precisely, if D is admissible, then the stability subgroup D_x for the transpose action of D on $\mathbb{R}^n$ must be compact for a.e. $x \in \mathbb{R}^n$; moreover, if Δ is the modular function of D, there must exist an $a \in D$ such that $|\det a| \neq \Delta(a)$. Conversely, if the last condition holds and for a.e. $x \in \mathbb{R}^n$ there exists an $\epsilon > 0$ for which the ϵ-stabilizer D_x^ϵ is compact, then D is admissible. Numerous examples are given of both admissible and non-admissible groups.

1 Introduction.

In order to introduce the wavelets we shall consider and the groups associated with them, it is useful to begin with some one dimensional results A.P. Calderón introduced in 1964, [C]. Let G be the *affine group* associated with $\mathbb{R}$. That is, G consists of all $(a, b) \in \mathbb{R} \times \mathbb{R}$, $a \neq 0$, with the group operation

$$(c, d) \cdot (a, b) = (ca, b + \frac{d}{a}).$$

This operation is consistent with the action of $s = (a, b) \in G$ on $x \in \mathbb{R}$ given by $s(x) = a(x + b)$. For $\psi \in L^2(\mathbb{R})$ let

$$(T_s\psi)(x) = \frac{1}{\sqrt{|a|}}\,\psi(\frac{x}{a} - b) = \frac{1}{\sqrt{|a|}}\,\psi(s^{-1}(x)) \equiv \psi_{a,b}(x).$$

Then $s \mapsto T_s$ is a unitary representation of G acting on $L^2(\mathbb{R})$. The mapping W_ψ, taking $f \in L^2(\mathbb{R})$ into the bounded function on G defined by

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$$(W_\psi f)(s) = \int_{\mathbb{R}} f(x)\, \overline{(T_s\, \psi)(x)}\, dx = \langle f, \psi_{a,b} \rangle$$

is often referred to as the *(continuous) wavelet transform* induced by ψ. As will later become clear, the term "continuous" really refers to the continuity of the *translations* (by $b \in \mathbb{R}$), rather than to the dilations a.

A goal in wavelet theory is to find a condition on ψ such that

$$||f||_2^2 = \int_G |(W_\psi\, f)(s)|^2 d\,\lambda(s) = \int_G |\langle f, \psi_{a,b} \rangle|^2 d\,\lambda(a,b) \tag{1.1}$$

for all $f \in L^2(\mathbb{R})$, where λ is left Haar measure on G (that is, up to a multiplicative constant, $d\,\lambda(a,b) = \frac{da db}{|a|}$). By the polarization identity applied to both sides of (1.1) we obtain the equality

$$\langle f, g \rangle = \int_G \langle f, \psi_{a,b} \rangle \langle \psi_{a,b}, g \rangle d\,\lambda(a,b)$$

for all f and g in $L^2(\mathbb{R})$ which, in turn, gives us a weak-topology interpretation of the *reproducing formula*

$$f(x) = \int_G \langle f, \psi_s \rangle \psi_s(x) d\,\lambda(s) = \int_G (W_\psi f)(s)(T_s\, \psi)(x) d\lambda(s). \tag{1.2}$$

This equality, often referred to as the Calderón reproducing formula, is valid for all $f \in L^2(\mathbb{R})$ if and only if ψ satisfies the Calderón *admissibility condition*,

$$\int_{\mathbb{R}\setminus\{0\}} |\widehat{\psi}(a)|^2 \frac{da}{|a|} = 1. \tag{1.3}$$

Since $da/|a|$ is the Haar measure of the multiplicative non-zero reals we can also re-write (1.3) in the form

$$\int_{\mathbb{R}\setminus\{0\}} |\widehat{\psi}(xa)|^2\, \frac{da}{|a|} = 1 \tag{1.3$'$}$$

for $x \in \mathbb{R}\setminus\{0\}$. We shall see shortly why this is a useful observation. The facts, as well as a higher dimensional version, were obtained by Calderón in [C].

The last two authors of the current paper proposed higher dimensional versions of these facts in [WW]. Let us describe these versions. We begin by considering the *Full Affine Group of Motions on* $\mathbb{R}^n$, $G^{\#}$: It consists of all pairs $(a, b) \in GL(n, \mathbb{R}) \times \mathbb{R}^n$ (endowed with the product topology) together with the operation

$$(\alpha, \beta) \cdot (a, b) = (\alpha\, a, a^{-1}\, \beta + b). \tag{1.4}$$

Let us agree that if c is an $n \times n$ real matrix and $y \in \mathbb{R}^n$, then cy is the column vector obtained by the usual matrix multiplication of c by y regarded as a column vector; while yc denotes the product of the $1 \times n$ matrix y with the $n \times n$ matrix c. The multiplication defined by (1.4) is consistent with the action $x \longrightarrow a(x + b)$ that the elements $(a, b) \in G^{\#}$ define on $\mathbb{R}^n$. We consider a class of subgroups, $\{G\}$, of $G^{\#}$ of the form

$$G = \{(a, b) \in G^{\#} : a \in D, b \in \mathbb{R}^n\},$$

where D is a closed subgroup of $GL(n, \mathbb{R})$. We identify D with the subgroup $\{(a, b) \in G : a \in D, b = 0\}$ and refer to it as the *dilation subgroup* of G; $\mathcal{N} = \{(a, b) \in G : a = I, b \in \mathbb{R}^n\}$ is called the *translation subgroup of* G. G is the semidirect product, $D \ltimes \mathcal{N}$, of D and $\mathcal{N}$. If μ is a left Haar measure for D, then $d\lambda(a, b) = d\,\mu(a)db$ is the element of a left Haar measure for G.

In complete analogy with the one dimensional case described earlier, we introduce the unitary representation T of G acting on $L^2(\mathbb{R}^n)$ by letting

$$(T_{(a,b)}\psi)(x) = |\det a|^{-1/2}\, \psi(a^{-1}x - b) \equiv \psi_{a,b}(x)$$

for $(a, b) \in G$ and $\psi \in L^2(\mathbb{R}^n)$. If we use the form of the Fourier transform

$$\widehat{f}(x) = \int_{\mathbb{R}^n} f(y) e^{-2\,\pi\, ix\cdot y} dy$$

when $f \in L^1(\mathbb{R}^n)$, then

$$(T_{(a,b)}\psi)^{\wedge}(x) = |\det a|^{1/2}\, \widehat{\psi}(xa) e^{-2\,\pi\, ix\cdot ab}. \tag{1.5}$$

The (*continuous*) *wavelet transform* W_ψ induced by ψ (and the group D) is defined by

$$(W_\psi f)(a, b) = \langle f, \psi_{a,b}\rangle = \int_{\mathbb{R}^n} f(y) \overline{\psi(a^{-1}y - b)}\, \frac{dy}{\sqrt{|\det a|}}$$

whenever $f \in L^2(\mathbb{R}^n)$ and $(a, b) \in G$. The adjective "continuous" refers to the continuity of the translation group, consisting of all $b \in \mathbb{R}^n$. The dilation group D, in contrast, is permitted to be discrete.

One goal of [WW] was to find an admissibility condition for ψ that guarantees the general form of the Calderón reproducing formula:

$$||f||_2^2 = \int_G |(W_\psi f)(a,b)|^2 \, d\lambda(a,b) = \int_G |\langle f, \psi_{a,b}\rangle|^2 d\,\lambda\,(a,b) \tag{1.6}$$

for all $f \in L^2(\mathbb{R}^n)$; or similarly to (1.2)

$$f(x) = \int_G \langle f, \psi_{a,b}\rangle \, \psi_{a,b}(x) d\,\lambda(a,b) \tag{1.6$'$}$$

(interpreted in the "weak sense").

Theorem (1.7) *(The admissibility condition). Equality (1.6) is valid for all $f \in L^2(\mathbb{R}^n)$ if and only if*

$$\Delta_\psi\,(x) \equiv \int_D |\widehat{\psi}(xa)|^2 d\mu(a) = 1 \tag{1.8}$$

for a.e. $x \in \mathbb{R}^n$.

This admissibility condition, which is a most natural extension of (1.3′), is proved in [WW] (and see the "formal" proof in the appendix to this paper). Also given in [WW] are some examples of dilation groups D for which there exist $\psi \in L^2(\mathbb{R})$ satisfying (1.8) and examples for which no such ψ exist. We shall present further examples below.

Let us state the major purpose of this work: we want to identify all those groups D that are admissible, meaning they possess a wavelet satisfying (1.8):

Definition. We say that D is *admissible* if and only if there exists a Borel measurable $h \in L^1(\mathbb{R}^n)$ such that $h \geq 0$ and

$$\int_D h(xa) d\,\mu(a) = 1 \quad \text{for a.e.} \quad x \in \mathbb{R}^n. \tag{1.9}$$

In terms of (1.8) we are, of course, letting h equal $|\widehat{\psi}|^2$; the desired wavelet can then be obtained by taking the inverse Fourier transform of $h^{1/2}$. We require Borel measurability in order to avoid certain joint measurability issues (in $G \times \mathbb{R}^n$) in some of our arguments later on. It is also useful to observe that our definition above and the arguments we use later on, apply more generally to σ-compact, locally compact groups D having a representation $\pi : D \longrightarrow GL(n, \mathbb{R})$ so that each $a \in D$ acts on row vectors $x \in \mathbb{R}$ by $x \mapsto x\,\pi(a)$. For simplicity we write xa instead of $x\,\pi(a)$ and, later on, "$\det a$" to denote the determinant of $\pi(a)$. Thus our definition of admissibility really applies to pairs (D, π).

Our principal result on admissibility involves the notion of the ϵ-*stabilizer* of $x \in \mathbb{R}^n$, defined as the set

$$D_x^\epsilon = \{a \in D : |xa - x| \leq \epsilon\}$$

for each $\epsilon \geq 0$. The set $D_x \equiv D_x^0 = \{a \in D : xa = x\}$ is referred to as the *stabilizer* of x. The modular function Δ on D (the function satisfying $\mu(Ea) = \Delta(a)\mu(E)$ for all μ-measurable $E \subset D$ and $a \in D$) also plays an important role in our development. We will write $\Delta \equiv |\det|$ to mean that $\Delta(a) = |\det a|$ $(= |\det \pi(a)|)$ for all $a \in D$. It is clear that $D_x = \cap_{\epsilon>0} D_x^\epsilon$, $D_x^{\epsilon_1} \subset D_x^{\epsilon_2}$ when $\epsilon_1 \leq \epsilon_2$. Another notion that is important to our study is the *orbit of* $x \in \mathbb{R}^n$ *produced by* D,

$$\mathcal{O}_x = xD = \{xa : a \in D\}.$$

We can now state our main result:

Theorem (1.8)

(a) If D is admissible, then $\Delta \not\equiv |\det|$ and the stabilizer of x is compact for a.e. $x \in \mathbb{R}^n$.

(b) If $\Delta \not\equiv |\det|$ and for a.e. $x \in \mathbb{R}^n$ there exists an $\epsilon > 0$ such that the ϵ-stabilizer of x is compact, then D is admissible.

Unfortunately, this result "just fails" to be a characterization of admissibility. Later on, in the observation before (2.11), we shall cite a result that "comes close" to showing that admissibility implies the compactness of D_x^ϵ for some $\epsilon = \epsilon(x) > 0$ for a.e. x. We hope that this "gap" can be closed, so that one can obtain a full characterization.

This theorem is quite useful for determining the admissibility of particular groups D. For example it is clear that no compact group D can be admissible since in this case $\Delta \equiv |\det| \equiv 1$. (In [WW] we presented a trivial direct argument that the compact group $S0(2)$ is not admissible.) We will discuss several groups and families of groups in the final section.

Before proving the thoerem we will illustrate various concepts by examining a simple example of an admissible group, the *first Galilei group*

$$D = \left\{ a = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} : \alpha \neq 0, \alpha, \beta \in \mathbb{R} \right\}.$$

The left Haar measure for D is $d\,\mu(a) = \alpha^{-2} d\,\alpha d\,\beta$ and the modular function is $\Delta(a) = \frac{1}{|\alpha|} = \frac{1}{|\det a|}$. The admissibility of D follows directly (without using our theorem) from the fact that if $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 \neq 0$, then

$$\int_D h(xa) d\,\mu(a) = \int_D h((1,0)a) d\,\mu(a)$$

for any h such that $x_1^{-2} h(x_1, x_2)$ is integrable on $\mathbb{R}^2$. Thus, all we need to do is normalize such a non-negative h in order to satisfy (1.9).

We end this section with a description of the orbits and stabilizers when D is the first Galilei group: if $x = (x_1, x_2)$

$$\mathcal{O}_x = \begin{cases} \mathbb{R}^2 \backslash (\{0\} \times \mathbb{R}) & \text{if} \quad x_1 \neq 0 \\ \{x\} = \{(0, x_2)\} & \text{if} \quad x_1 = 0 \end{cases} . \tag{1.10}$$

The ϵ-stabilizer is $D_x^\epsilon = D$ if $x_1 = 0$ and $\epsilon \geq 0$. If $x_1 \neq 0$ and $\epsilon > 0$, then

$$D_x^\epsilon = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in D : (\alpha - 1)^2 + \beta^2 \leq \left(\frac{\epsilon}{x_1} \right)^2 \right\};$$

while, if $x_1 \neq 0$ and $\epsilon = 0$, then $D_x^0 = D_x = \{I\} = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})\}$. Observe that D_x^ϵ is compact if $\epsilon < |x_1|$; on the other hand, when $\epsilon \geq |x_1| > 0$, D_x^ϵ is not compact (the sequence $a_n = (\begin{smallmatrix} 2^{-n} & 0 \\ 0 & 1 \end{smallmatrix}), n \in \mathbb{N}$, belongs to D_x^ϵ in this case but no subsequence converges). In any event, part (b) of our Theorem guarantees admissibility of the Galilei group D because $\Delta(a) = \frac{1}{|\alpha|} \not\equiv |\alpha| = |\det a|$, and for all (x_1, x_2) with $x_1 \neq 0$, the ϵ-stabilizer is compact when $\epsilon < |x_1|$.

2 Proof of the main theorem.

We shall break up the proof into four natural parts. Part (a) of the theorem consists of two implications; part (b) involves two hypotheses and, as we shall see, each plays a different role for achieving the desired conclusion. Hence, there are four arguments that make up our proof. Let us begin with the first part:

Proposition (2.1) *If D is admissible then $\Delta \not\equiv |\det|$.*

Proof Since D is admissible there exists a non-negative, Borel measurable $h \in L^1(\mathbb{R}^n)$ such that (1.9) is satisfied for almost every $x \in \mathbb{R}^n$. We first show that

$$\int_{\mathbb{R}^n} h(x)dx > 0. \tag{2.2}$$

To see this, let $B(0,1)$ be the unit ball about the origin in $\mathbb{R}^n$. Then,

$$|B(0,1)| = \int_{B(0,1)} \int_D h(xa)d\,\mu(a)dx = \int_D \{\int_{B(0,1)a} h(y)dy\}\,|\det a|^{-1}d\mu(a).$$

In particular, $h(x)$ cannot be zero a.e. and (2.2) follows.

Let $f(x) = 2^n h(2x)$. Then,

$$\int_{\mathbb{R}^n} h(x)dx = \int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \{\int_D h(xa)d\mu(a)\} f(x)dx =$$

$$\int_D \{\int_{\mathbb{R}^n} f(xa^{-1})h(x)|\det a|^{-1}dx\}d\,\mu(a) = \int_{\mathbb{R}^n} h(x) \int_D f(xb)\frac{|\det b|}{\Delta(b)}\,d\,\mu(b)dx,$$

where the last equality is obtained by the change of variable $a = b^{-1}$ and the modular function property $d\,\mu(b^{-1}) = d\mu(b)/\Delta(b)$. Thus, if $\Delta \equiv |\det|$,

$$\begin{aligned}\int_{\mathbb{R}^n} h(x)dx &= \int_{\mathbb{R}^n} h(x) \int_D f(xb)d\,\mu(b)dx \\ &= \int_{\mathbb{R}^n} h(x)2^n\{\int_D h(2xb)d\,\mu(b)\}dx = 2^n \int_{\mathbb{R}^n} h(x)dx\end{aligned}$$

since the expression within brackets is 1 almost everywhere. But this is impossible since $0 < \int_{\mathbb{R}^n} h(x)dx < \infty$. □

Proposition (2.3) *If D is admissible, then D_x is compact for almost every $x \in \mathbb{R}^n$.*

Proof. Again, as in the last proof, we choose an h that satisfies (1.9). Let us fix an $x \in \mathbb{R}^n$ for which (1.9) is true. Let C be *any* compact subset of D and $b_1 \in D_x$. If we assume that D_x is not compact (though it is closed), then there exists $b_2 \in D_x$ such that $b_2 \notin b_1CC^{-1}$ since this last set is compact (otherwise D_x would be a closed subset of a compact set, contradicting the assumption that it is not compact). Continuing this process we obtain a collection of compact sets $\{b_kC\}$ with $b_k \in D_x \backslash \{b_1CC^{-1} \cup \ldots \cup b_{k-1}CC^{-1}\}$, $k \geq 2$. This collection is mutually disjoint since, if $1 \leq l < k$ and $b_lC \cap b_kC \neq \phi$, then $b_lc_1 = b_kc_2$ for $c_1, c_2 \in C$ and this would imply $b_k = b_lc_1c_2^{-1}$, contrary to the construction.

We now use (1.9), this disjointness property and the fact that $xb_k = x$ for all $k \in \mathbb{N}$ to obtain $1 = \int_D h(xa)d\,\mu(a) \geq \sum_{k=1}^{\infty} \int_{b_kC} h(xa)d\,\mu(a) =$ (since μ is left Haar measure on D) $\sum_{k=1}^{\infty} \int_C h(xb_ka)d\mu(a) = \sum_{k=1}^{\infty} \int_C h(xa)d\,\mu(a)$. Since each of the last summands is the same, it follows that $\int_C h(xa)d\,\mu(a) = 0$. But C is an arbitrary compact subset of D. But the latter is σ-compact and, thus, a countable union of compact sets. It then follows that

$$\int_D h(xa)d\,\mu(a)$$

must be 0 and not 1, as assumed. $\square$

Propositions (2.1) and (2.3) give us part (a) of our main theorem. Let us now turn our attention to part (b). As we stated before, two assumptions are involved; the first is that the set

$$\Omega_0 = \{x \in \mathbb{R}^n : D_x^{\epsilon} \quad \text{is non-compact for all } \epsilon > 0\}$$

has measure zero. We will show that this hypothesis allows us to define a Borel measurable $g : \mathbb{R}^n \longrightarrow [0, \infty)$ such that

$$\int_D g(xa)d\,\mu(a) = 1 \quad \text{if} \quad x \notin \Omega_0. \tag{2.4}$$

The second hypothesis, $\Delta \not\equiv |\det|$, will be used to modify g into a function $h \in L^1(\mathbb{R}^n)$ satisfying (1.9).

We introduce a class of functions associated with D that are useful for constructing the g we described above. We call these functions the *orbit density functions* since they measure how much of an orbit lies within each ball. Given an open ball $B \subset \mathbb{R}^n$ we define $f_B : \mathbb{R}^n \to [0, \infty]$ by

$$f_B(x) = \mu(\{a \in D : xa \in \overline{B}\}) = \int_D \chi_{\overline{B}}(xa)d\,\mu(a).$$

We list some properties of f_B that are immediate:

Lemma (2.5)

i. $f_B(x) = f_B(xa)$ *for all* $a \in D$*; that is,* f_B *is constant on orbits.*

ii. *If $B(x, \epsilon)$ is the open ball of radius $\epsilon > 0$ about x, then $f_{B(x,\epsilon)}(x) = \mu(D_x^\epsilon)$.*

iii. *If $\overline{B} \cap \mathcal{O}_x = \phi$ then $f_B(x) = 0$.*

iv. *If $B \cap \mathcal{O}_x \neq \phi$ then $f_B(x) > 0$.*

v. *If $B \subset B'$, then $f_B \leq f_{B'}$.*

vi. *If D_x^ϵ is compact then $f_B(x) < \infty$ for all $B \subset B(x, \epsilon)$.*

The following result (relating Ω_0 to the orbit density functions) will help us construct the function g.

Lemma (2.6) *The set Ω_0 is equal to the set*

$$\widetilde{\Omega}_0 = \{x \in \mathbb{R}^n : f_B(x) = \infty \quad \textit{for all} \quad B \quad \textit{such that} \quad B \cap \mathcal{O}_x \neq \phi\}.$$

Proof. We begin by showing that $\widetilde{\Omega}_0 \subset \Omega_0$. Suppose $x \notin \Omega_0$ so that D_x^ϵ is compact for some $\epsilon > 0$. Then, by Lemma (2.5) part (ii), $f_{B(x,\epsilon)}(x) = \mu(D_x^\epsilon) < \infty$. Moreover, $B(x, \epsilon) \cap \mathcal{O}_x \neq \phi$. Thus, $x \notin \widetilde{\Omega}_0$ and so $\Omega_0^c \subset \widetilde{\Omega}_0^c$, or $\widetilde{\Omega}_0 \subset \Omega_0$.

Now suppose $x \notin \widetilde{\Omega}_0$; we show $x \notin \Omega_0$. Since $x \notin \widetilde{\Omega}_0$, there exists an open ball $B \subset \mathbb{R}^n$ such that $B \cap \mathcal{O}_x \neq \phi$ and $f_B(x) < \infty$. We first show that there exists $\epsilon > 0$ such that $\mu(D_x^\epsilon) < \infty$. Since $B \cap \mathcal{O}_x \neq \phi$ there exists $a \in D$ such that $xa \in B$ or, equivalently, $x \in Ba^{-1}$. Since B is open, there exists $\epsilon > 0$ such that $\overline{B(x, \epsilon)} \subset Ba^{-1}$. It follows that

$$D_x^\epsilon a = \{ba : xba \in \overline{B(x, \epsilon)}a\} \subset \{ba : xba \in \overline{B}\} = \{c \in D : xc \in \overline{B}\}.$$

Thus, using a basic property of the modular function,

$$\mu(D_x^\epsilon) = \Delta(a^{-1})\mu(D_x^\epsilon a) \leq \Delta(a^{-1})\mu(\{c \in D : xc \in \overline{B}\}) = \Delta(a^{-1}) f_B(x) < \infty.$$

We shall now use this fact to show that there exists an $\epsilon_1 > 0$ such that $D_x^{\epsilon_1}$ is compact. Choose a compact $C \subset D$ such that C contains a (small) open set about the identity, I, in D. Using the $\epsilon > 0$ we just found (so that $\mu(D_x^\epsilon) < \infty$) and by our choice of C we have

$$\mu(\{c \in C : xc \in \overline{B(x, \frac{\epsilon}{2})}\}) > 0. \tag{2.7}$$

Moreover, $\gamma = \max_{c\in C} ||c|| \geq ||I|| = 1$. ($\gamma$ is, of course, finite. Also, if we are dealing with the pair (D, π), then $||c||$ means $||\pi\, c||$.) Let $\epsilon_1 = \epsilon/2\,\gamma$ and assume $D_x^{\epsilon_1}$ is not compact. As we did in the proof of Proposition (2.3), we construct a sequence $\{b_k\} \subset D_x^{\epsilon_1}$ such that the sets $\{b_k C\}$ are pairwise disjoint. If $c \in C$ and $xc \in \overline{B(x, \frac{\epsilon}{2})}$, then $|x - xb_k c| \leq |x - xc| + |xc - xb_k c| \leq \frac{\epsilon}{2} + |x - xb_k|\ ||c|| \leq \frac{\epsilon}{2} + \epsilon_1\,\gamma = \epsilon$. Hence, for all $k \in \mathbb{N}$,

$$\{c \in C : xc \in \overline{B(x, \frac{\epsilon}{2})}\} \subset \{c \in C : xb_k c \in \overline{B(x, \epsilon)}\}.$$

We have, therefore,

$$\infty > \mu(D_x^\epsilon) \geq \mu(\bigcup_{k=1}^{\infty} D_x^\epsilon \cap b_k C) = \sum_{k=1}^{\infty} \mu(b_k\{c \in C : xb_k c \in \overline{B(x,\epsilon)}\}) \geq$$

$$\sum_{k=1}^{\infty} \mu(\{c \in C : xc \in \overline{B(x, \frac{\epsilon}{2})}\}).$$

But the summands of this last sum are independent of k and all equal a positive number by (2.7). This contradicts the fact that $\mu(D_x^\epsilon) < \infty$. Thus $D_x^{\epsilon_1}$ is compact. This shows that $x \notin \Omega_0$. Hence, $\widetilde{\Omega}_0^c \subset \Omega_0^c$ and, together with $\widetilde{\Omega}_0 \subset \Omega_0$ we see that the sets Ω_0 and $\widetilde{\Omega}_0$ are equal. □

We are now ready to establish part (b) of our main theorem. As in part (a) we do this by proving two propositions.

Proposition (2.8) *If the Lebesgue measure of* Ω_0 *is* 0 *then there exists a Borel measurable function* $g : \mathbb{R}^n \to [0, \infty)$ *such that* $\int_D g(xa)d\,\mu(a) = 1$ *for a.e.* $x \in \mathbb{R}^n$.

Remark. It is clear from (2.5) (i) (invariance of the orbit density function) and the definition of $\widetilde{\Omega}_0$ that if $x \in \widetilde{\Omega}_0$ then $\mathcal{O}_x \subset \widetilde{\Omega}_0$. A natural first attempt to produce the function g is to define it to be the ratio $\chi_{\overline{B}}(x)/f_B(x)$, at least for those x such that $f_B(x) < \infty$ (which can be done if $x \notin \widetilde{\Omega}_0$) and $0 < f_B(x)$ (which would require a choice of B). If this could be done we would have

$$\int_D g(xa)d\,\mu(a) = \int_D \frac{\chi_{\overline{B}}(xa)}{f_B(x)}\,d\,\mu(a) = 1 \qquad (2.9)$$

(recall that f_B is constant on $\mathcal{O}_x$). Our proof consists of making a proper use of this idea in a way that captures almost every x.

Proof. Let $\mathcal{B} = \{B_j\}, j \in \mathbb{N}$, be an enumeration of the balls in $\mathbb{R}^n$ having rational centers and positive rational radii. Let $f_j = f_{B_j}$. We claim that

$$\mathbb{R}^n = \widetilde{\Omega}_0 \cup (\bigcup_{j\geq 1}\{x \in \mathbb{R}^n : 0 < f_j(x) < \infty\}). \tag{2.10}$$

To see this, suppose $x \notin \widetilde{\Omega}_0$. Then there exists an open ball B such that $B \cap \mathcal{O}_x \neq \phi$ and $f_B(x) < \infty$. Hence, there exists $a_0 \in D$ such that $xa_0 \in B$. Since B is open we can find j such that $xa_0 \in B_j \subset B$. But this means that

$$B_j \cap \mathcal{O}_x \neq \phi.$$

By (2.5) (iv) this implies that $f_j(x) > 0$. We can now invoke (2.5) (v) and the fact that $B_j \subset B$ to also conclude that $f_j(x) < \infty$. Thus,

$$x \in \{y \in \mathbb{R}^n : 0 < f_j(y) < \infty\}$$

and (2.10) is established.

It follows that

$$\Omega_1 = \{x \in \mathbb{R}^n : 0 < f_1(x) < \infty\}, \Omega_2 = \{x \in \mathbb{R}^n : 0 < f_2(x) < \infty\}\backslash\Omega_1, \ldots,$$
$$\Omega_j = \{x \in \mathbb{R}^n : 0 < f_j(x) < \infty\}\backslash\{\Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_{j-1}\}, \ldots$$

form a disjoint collection of Borel sets such that $\mathbb{R}^n\backslash \cup_{j=1}^{\infty} \Omega_j$ has measure 0 (it is a subset of $\widetilde{\Omega}_0 = \Omega_0$). Let us define g to be zero on this set of measure 0 and, if $x \in \cup_{j=1}^{\infty} \Omega_j$, so that $x \in \Omega_j$ for a unique $j \geq 1$, we let

$$g(x) = \frac{\chi_{\overline{B_j}}(x)}{f_{B_j}(x)}.$$

That is,

$$g(x) = \sum_{j=1}^{\infty} \chi_{\Omega_j}(x)\,\frac{\chi_{\overline{B_j}}(x)}{f_{B_j}(x)}$$

for $x \in \cup_{j=1}^{\infty} \Omega_j$, and $g(x) = 0$ on the complement of this union. Using the argument we presented for equality (2.9) we obtain the desired property for g and this establishes (2.8) (we remind the reader that the sets Ω_j are orbit invariant: $\Omega_j a = \Omega_j$ for all $a \in D$). □

Observations. Let us comment about certain subtleties involved in these arguments and ideas. The set $\widetilde{\Omega}_0 = \Omega_0$ might not be disjoint from the sets $\Omega_j, j \geq 1$ because $\mathcal{O}_x$ might intersect $\overline{B}_j$ without intersecting B_j. By considering $\mathbb{R}^n \backslash \{\cup_{j\geq 1} \Omega_j\} (\subset \Omega_0)$ we avoid this problem and, thus, the function g is well defined and Borel measurable. We also point out to the reader that the use of the open ball B (as in the definition of $\widetilde{\Omega}_0$) and its closure $\overline{B}$ (as in the definition of f_B) are important features of our argument.

In the comments made after the statement of the main theorem (1.8) we stated that we feel that the sufficient condition should also be necessary (thus giving a characterization of admissibility); more specifically, we suspect admissibility should imply that a.e. x has a compact ϵ-stabilizer for some $\epsilon > 0$. In this dierection, we can show that if D is admissible then for a.e. $x \in \mathbb{R}^n$ and for all $\delta > 0$ there exists $\epsilon > 0$ such that $D_x^\epsilon \cap \{|\det| > \delta\}$ is compact. The proof is rather complicated and we do not include it here.

We now turn to the construction of the function $h \in L^1(\mathbb{R}^n)$ that satisfies (1.9); we restate and then prove part (b) of our theorem:

Proposition (2.11) *If the Lebesgue measure of Ω_0 is zero and $\Delta \not\equiv |\det|$, then D is admissible.*

Proof. We need to construct a non-negative integrable function h on $\mathbb{R}^n$, satisfying (1.9). We do this by "translating g along orbits" without affecting equality (1.9) which, by Proposition (2.8), is known to be satisfied by g. Let

$$E_{jk} = \{x \in \mathbb{R}^n : 2^j < |x| \leq 2^{j+1} \quad \text{and} \quad 2^k < g(x) \leq 2^{k+1}\},$$

for $j, k \in \mathbb{Z}$, and put $g_{jk} = g\, \chi_{E_{jk}}$, so that

$$g = \sum_{j,k\in\mathbb{Z}} g_{jk} \quad \text{for} \quad x \neq 0. \tag{2.11}$$

We are assuming the existence of a $b \in D$ such that $\Delta(b) \neq |\det b|$. Thus, for each $j, k \in \mathbb{Z}$ we can find $l(j,k) \in \mathbb{Z}$ such that

$$\left(\frac{\Delta(b)}{|\det b|}\right)^{l(j,k)} 2^{k+1}|E_{jk}| < 2^{-|j|-|k|}.$$

We now define h by letting

$$h(x) = \sum_{j,k\in\mathbb{Z}} h_{jk}(x),$$

where

$$h_{jk}(x) = \Delta(b)^{l(j,k)} g_{jk}(xb^{l(j,k)}).$$

Observe that $0 \le h \in L^1(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} h(x)dx = \sum_{j,k\in\mathbb{Z}} \Delta(b)^{l(j,k)} \int_{\mathbb{R}^n} g_{jk}(xb^{l(j,k)})dx =$$

$$\sum_{j,k\in\mathbb{Z}} \Delta(b)^{l(j,k)} |\det b|^{-l(j,k)} \int_{\mathbb{R}^n} g_{jk}(x)dx \le$$

$$\sum_{j,k\in\mathbb{Z}} \left(\frac{\Delta(b)}{|\det b|}\right)^{l(j,k)} 2^{k+1}|E_{jk}| \le \sum_{j,k\in\mathbb{Z}} 2^{-|j|-|k|} < \infty.$$

Next we show that h (obviously Borel measurable) satisfies (1.9):

$$\int_D h(xa)d\,\mu(a) = \sum_{j,k\in\mathbb{Z}} \Delta(b)^{l(j,k)} \int_D g_{jk}(xab^{l(j,k)})d\,\mu(a) =$$

$$\sum_{j,k\in\mathbb{Z}} \int_D g_{jk}(xa)d\,\mu(a) = \int_D g(xa)d\,\mu(a) = 1$$

for a.e. x (the second equality is a consequence of the modular function property

$$\Delta(b) \int_D f(ab)d\,\mu(a) = \int_D f(a)d\,\mu(a),$$

valid whenever f is μ-integrable).

This proves Proposition (2.11) and concludes our proof of the main theorem. □

3 Applications of the Main Theorem, and Examples.

Perhaps a most natural version of the affine group acting on $\mathbb{R}^n$ is the one in which the dilation group D is a one-parameter group. That is,

$$D = \{a(t) = e^{tL} : t \in \mathbb{R}\}, \tag{3.1}$$

where L is an $n \times n$ matrix with real coefficients. Clearly, $\det a(t) = e^{t(\operatorname{trace} L)}$ and D is Abelian. Consequently, the modular function Δ is identically 1. It follows from our main theorem that, if D is admissible, then $|\det| \not\equiv 1$ and so $\operatorname{trace} L \neq 0$. We claim the converse is also true:

Corollary 3.2 *A one parameter group D (that is, D has the form (3.1)) is admissible if and only if* $(\operatorname{trace} L) \neq 0$.

Proof. We have just seen that admissibility implies $(\operatorname{trace} L) \neq 0$. We also have just observed that this last condition implies $\Delta \not\equiv |\det|$; thus, by part (b) of our main theorem, we have only to show that $\operatorname{trace} L \neq 0$ also implies that for a.e. $x \in \mathbb{R}^n$ there exists $\epsilon > 0$ such that D_x^ϵ is compact.

Let $L = PJP^{-1}$, where J is the (complex) Jordan form of L. Also, let us agree that the "ones" in this Jordan form are *above* the diagonal entries and, thus, the upper left entry, λ, in the block involving this proper value produces a column vector for which λ is the only possible non-zero coordinate. Of course, the proper values of L (which are the same as those of J) need not be real; however, the fact that the entries of L are real and $(\operatorname{trace} L \neq 0$ does imply that at least one proper value λ has a non-zero real part. Choose k such that the (k, k) entry of J equals this λ and, also, is the upper left entry of the block associated with λ. Then the entries of the k^{th} column of e^{tJ} are all 0 except for the k^{th} one, which is $e^{\lambda t}$. If $y = (\ldots, y_k, \ldots) \in \mathbb{C}^n$ ($y_k = k^{th}$ component of y), then $y_k e^{t\lambda}$ is the k^{th} component of ye^{tJ}. If $y_k \neq 0$, then since $\mathcal{R}e\,\lambda \neq 0$,

$$\lim_{t\to-\infty} |y_k e^{t\lambda}| \quad \text{and} \quad \lim_{t\to\infty} |y_k e^{t\lambda}| \tag{3.3}$$

are either 0 or ∞, depending on the sign of $\mathcal{R}e\,\lambda$. Let $y = xP$ with $x \in \mathbb{R}^n$ so chosen that $y_k \neq 0$ (since P is invertible, the solutions x of $y_k = (xP)_k = 0$ lie in a hyperplane in $\mathbb{R}^n$ which must have measure 0). It follows from (3.3) that the distance between $y = xP$ and $ye^{tJ} = xPe^{tJ}$ exceeds a fixed positive quantity if $|t|$ is large. The same must be true for $xe^{tL} = xPe^{tJ}P^{-1}$ and $x = (xP)P^{-1}$. But this means that for $\epsilon > 0$ sufficiently small, $D_x^\epsilon = \{a(t) = e^{tL} : |xa(t) - x| \leq \epsilon\}$ involves only a bounded set of $t \in \mathbb{R}$. This and the fact that D_x^ϵ is closed imply that D_x^ϵ is compact. □

The above argument can be adapted to one parameter groups where the parameter domain is $\mathbb{Z}$ instead of $\mathbb{R}$. We can conclude, therefore, that $D = \{a(k) = e^{kL} : k \in \mathbb{Z}\}$ is admissible if and only if $(\operatorname{trace} L) \neq 0$. This collection of groups includes the dilation groups

$$D = \{M^k : k \in \mathbb{Z}\}, \tag{3.4}$$

where M is an expanding matrix in $GL(n,\mathbb{R})$ (that is, all the proper values, λ, of M satisfy $|\lambda| > 1$).

These groups are of special interest because they are involved with the *discrete* wavelets of the form

$$\psi_{jk}(x) = |\det M|^{1/2}\psi(M^j x - k),$$

where $j \in \mathbb{Z}$ and k ranges through a discrete lattice $\Gamma = B\mathbb{Z}^n$ ($B \in GL(n,\mathbb{R})$) (see [CCMW] for a complete characterization of these wavelets). Thus, we are provided with an example of a discrete dilation group *and* a discrete group of translations (see [WW] for a discussion of these groups and some of the features that distinguish these discrete wavelets from the continuous ones we study in this paper.)

This does raise the question of the characterization of the admissible discrete dilation groups that are not, simply, of the form (3.4), the integral powers of an expanding matrix. This study as well as the more intricate problem of the discretization of continuous translation subgroups will be presented in a following paper.

Let us now present a list of examples of groups D that illustrate how our main theorem can be applied.

Example 1.

$$D = \{a = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : x \neq 0, x, y \in \mathbb{R}\} \quad \textit{is admissible.} \tag{3.5}$$

To see this, we observe that left Haar measure is $d\,\mu_l(x,y) = x^{-2}dxdy$ while right Haar measure is $d\,\mu_r(x,y) = dxdy$. Consequently, D is not unimodular and, thus, $\Delta \not\equiv 1 \equiv |\det|$. It is also clear that if $u = (u_1, u_2)$ is a point of $\mathbb{R}^2$ with $u_1 \neq 0$, then the ϵ-stabilizer D_u^ϵ is compact for $\epsilon > 0$ sufficiently small. Thus, part (b) of our main theorem shows that D is admissible.

Example 2.

$$\text{Let } D = \{2^j a : j \in \mathbb{Z},\ a \text{ in a fixed compact subgroup of } GL(n,\mathbb{R})\}. \tag{3.6}$$

Then D is admissible.

Proof: Use part (b) of our main theorem. Note $\Delta \equiv 1 \neq 2^{nj} = |\det|$.

We remind the reader that, as we have observed above, a compact subgroup of $GL(n,\mathbb{R})$ is never admissible. This example provides us with a method of "creating" an admissible group from one that is not by multiplying the latter by the group $\{2^n I : n \in \mathbb{Z}\}$.

We should compare this example with the following one that was given to us by Hartmut Führ: Let

$$D = \{2^j a : j \in \mathbb{Z}, a \in SL(2,\mathbb{Z})\}. \tag{3.7}$$

Then, D is not admissible. Moreover, $\Delta \not\equiv |\det|$ and for a.e. $x \in \mathbb{R}^2$ the stabilizer is compact.

This example is particularly relevant to us since it furnishes a non-admissible group D that satisfies the necessary conditions in part (a) of our main theorem. Thus, our necessary conditions are not sufficient. It is also of interest because of our observation preceding Proposition (2.11) that admissibility implies that for a.e. x and for all $\delta > 0$ there exists $\epsilon > 0$ such that $D_x^\epsilon \cap \{|\det| > \delta\}$ is compact. In the example given in (3.7) this property is not true. The group $SL(2,\mathbb{Z})$ enjoys the following "ergodic" property: if U is an open subset of $\mathbb{R}^2$ and $x = (x_1, x_2)$ is such that x_1/x_2 is irrational then there exists $a \in SL(2,\mathbb{Z})$ such that $xa \in U$ (see [Z], (2.19)). In particular, $\{y \in \mathbb{R}^2 : |y - 2^{-j}x| < 2^{-j}\epsilon\}, j \in \mathbb{Z}$, is open; thus, there exists $a \in SL(2,\mathbb{Z})$ such that $|xa - 2^{-j}x| < 2^{-j}\epsilon$. Equivalently, $|2^j xa - x| < \epsilon$ and this shows that $D_x^\epsilon \cap \{|\det| > \delta\}$ cannot be compact for any $\epsilon, \delta > 0$ since $2^j xa \in D_x^\epsilon$ with j arbitrarily large.

Thus, this example is also interesting since it is a discrete group that "just fails" to be admissible.

Example 3.

$$D = \{\begin{pmatrix} 2^j & 0 \\ 0 & 2^l \end{pmatrix} : j, l \in \mathbb{Z}\} \quad \textit{is an admissible discrete group.} \tag{3.8}$$

This is a very simple example of a discrete admissible group that is not cyclic (as in (3.4)).

Example 4. Generalizations of the Galilei groups are considered in Physics; their admissibilities and the associated reproduction formulae are of particular interest in the study of "coherent states" (see [AAG]). These groups, in particular, have the form of the Galilei group we introduced in §1 except that α is an appropriate invertible $n \times n$ matrix and β is an $n \times 1$ column vector. Thus, these are subgroups of $GL(n+1,\mathbb{R})$. Their admissibility is a consequence of appropriate choices of subgroups of $GL(n,\mathbb{R})$ whose elements are the matrices α. Our main theorem is quite useful for determining their admissibility.

Example 5. Finally, let us observe that the Heisenberg groups, $GL(n,\mathbb{R})$, and $SL(n,\mathbb{R})$ are not admissible (the second because of non-compact stabilizers, the first and third since $|\det| \equiv 1 \equiv \Delta$).

We would like to comment briefly on the relationship between our results and those of several recent papers and manuscripts: [AAG], [Br], [BT], [DN], [F_1], [F_2], [F_3], [FM] and [IS]. The admissibility condition (1.8) is known to many researchers and the concept of admissibility we are considering is a special case of the widely studied notion of a square unitary representation as well as a special case of the notion of a coherent state discussed in [AAG]. In the articles by Führ and his collaborators there are several interesting results elucidating the "general harmonic analysis" results of Duflo and Moore [DM]. In [BT] and [DN] and the various articles by Führ interesting examples are given of representations of a group D that are admissible relative to an open set $U \subset \mathbb{R}^n$ (for this one requires that $xa \in U$ for $x \in U$ and $a \in D$, with (1.6) required to hold only for those $f \in L^2(\mathbb{R}^n)$ vanishing outside U; condition (1.8) must then hold for a.e. $x \in U$). In this setting, as well as more general settings (replacing $\mathbb{R}^n$ with a non-abelian Lie group) one can formulate an analog of part (a) in our main theorem as a necessary structural condition for a group to be admissible. Our main contribution is the sufficient condition (b) in our main theorem, in the standard wavelet context of reproducing function systems for $L^2(\mathbb{R}^n)$. No assumptions on irreducibility are needed, nor are we obliged to assume that D is a Lie group.

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Appendix. The following "formal" derivation of the general admissibility condition may be useful for understanding of concept.

Suppose the generalized Calderón reproducing formula (1.6′) holds pointwise at $x = 0$, so that

$$
\begin{aligned}
f(0) &= \int_G \langle f, \psi_{a,b}\rangle \psi_{a,b}(0)\, d\,\lambda(a,b) \\
&= \int_D \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{\psi(a^{-1}y - b)}\, \psi(-b) db\, |\det a|^{-1} f(y) dy\, d\,\mu(a) \\
&= \int_D \int_{\mathbb{R}^n} (\overline{\psi} * \psi_{-})(a^{-1}y)\, |\det a|^{-1} f(y) dy\, d\,\mu(a) \\
&= \int_D \int_{\mathbb{R}^n} (\overline{\psi} * \psi_{-})(y) f(ay) dy\, d\,\mu(a)
\end{aligned}
$$

(in order to pass from the second-to-last equation to the last equation we use $\psi_-(x) = \psi(-x)$, and replace $y \mapsto ay$). Now apply this formula to the function $f(y) = e^{2\pi i \xi y}$ (for $\xi \in \mathbb{R}^n$ a fixed row vector), obtaining

$$1 = \int_D (\overline{\psi} * \psi_-)^\wedge(-\xi a) d\mu(a) = \int_D |\widehat{\psi}(\xi a)|^2 \, d\mu(a),$$

which is the admissibility condition.

Of course the above argument is not rigorous, since we applied the reproducing formula pointwise, and to an f that is not square integrable.

References

[AAG] Ali, S.T., Antoine, J-P, and Gazeau, J-P. *Coherent States, Wavelets and Their Generalizations.* Springer-Verlag, New York, Inc.: 2000.

[BGZ] Bacry, H., Grossman, A., and Zak, J. *Geometry of Generalized Coherent States.* Group Theoretical Methods in Physics (Fourth International Colloquium, Nijmegen , 1975), Springer-Verlag, Berlin, Lecture Notes in Physics, 50: 249-268, 1978.

[BT] Bernier, D., and Taylor, K.F. *Wavelets from Square-integrable Representations.* SIAM J. Math. Anal., 27(2): 594-608, 1996.

[Br] Bruna, J. *Sampling in Complex and Harmonic Analysis.* Proc. of 3^{rd} Eur. Conf. in Mathematics, July, 2000.

[C] Calderón, A.P. *Intermediate Spaces and Interpolation, the Complex Method.* Stud. Math. 24, 113-190, 1964.

[Car] Carey, A.L. *Square Integrable Representations of Non-unimodular Groups.* Bull. Austral. Math. Soc., 15: 1-12, 1976.

[CCMW] Chui, C., Czaja, W., Maggioni, M. and Weiss, Guido. *Characterization of General Tight Wavelet Frames with Matrix Dilation and Tightness Preserving Oversampling.* To appear in J. Four. Anal. and Applications.

[Co] Corbett, J. *Coherent States on Kinematic Groups: The Study of Spatio-Temporal Wavelets and Their Applications to Motion Estimation.* Ph.D. Thesis, Washington University, 1999.

[DM] Duflo, M. and Moore, C.C. *On the Representation of a Nonunimodular Locally Compact Group.* J. Funct. Anal. 21, 209-243, 1976.

[DN] DeMari, F. and Nowak, K. *Reproducing Formulas and Commutative Operator Algebras Related to Affine Transformations of the Time-Frequency Plane.* Preprint.

[F_1] Führ, H. *Wavelet Frames and Admissibility in Higher Dimensions.* J. of Math. Phys., 37, 6353-6366, 1996.

[F_2] Führ, H. *Continuous Wavelet Transforms from Semidirect Products.* To appear in Revista Ciencias Matematicas.

[F_3] Führ, H. *Admissible vectors for the regular representations.* To appear in Proc. Amer. Math. Soc.

[FM] Führ, H and Mayer, M. *Continuous Wavelet Transforms from Cyclic Representations: A General Approach Using Plancherel Measure.* Preprint, 1999.

[HW] Heil, C.E. and Walnut, D.F. *Continuous and Discrete Wavelet Transforms.* SIAM Review, 31: 628-666, 1989.

[IS] Isham, C.J., and Klauder J.R. *Coherent States for n-dimensional Euclidean groups and their application.* J. of Math. Phys., 32, 607-620, 1991.

[K] Kalisa, C. *Etats Cohérents Affines: Canoniques, Galiléns et Relativistes.* Ph.D. Thesis, Université Louvaine, 1993.

[LP] Liu, H., and Peng, L. *Admissible Wavelets Associated with the Heisenberg Group.* Pacific Jour. of Math., 180: 101-123, 1997.

[WW] Weiss, Guido and Wilson, E.N. *The Mathematical Theory of Wavelets.* Proceedings of the NATO-ASI meeting "Harmonic Analysis 2000-A Celebration." Kluwer (Publisher), to appear in 2001.

[Z] Zimmer, R.J. *Ergodic Theory and Semisimple Groups.* Birkhäuser, Boston, 1980.

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A Unified Characterization of Reproducing Systems Generated by a Finite Family, II

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Abstract

By a "reproducing" method for $\mathcal{H} = L^2(\mathbb{R}^n)$ we mean the use of two countable families $\{e_\alpha : \alpha \in \mathcal{A}\}$, $\{f_\alpha : \alpha \in \mathcal{A}\}$, in $\mathcal{H}$, so that the first "analyzes" a function $h \in \mathcal{H}$ by forming the inner products $\{< h, e_\alpha >: \alpha \in \mathcal{A}\}$, and the second "reconstructs" h from this information: $h = \sum_{\alpha\in\mathcal{A}} < h, e_\alpha > f_\alpha$. A variety of such systems have been used successfully in both pure and applied mathematics. They have the following feature in common: they are generated by a single or a finite collection of functions by applying to the generators two countable families of operators that consist of two of the following three actions: dilations, modulations, and translations. The **Gabor systems**, for example, involve a countable collection of modulations and translations; the **affine systems** (that produce a variety of wavelets) involve translations and dilations. Considerable amount of research has been conducted in order to characterize those generators of such systems. In this paper we establish a result that "unifies" all of these characterizations by means of a relatively simple system of equalities. Such unification has been presented in a work by one of us. One of the novelties here is the use of a different approach that provides us with a considerably more general class of such reproducing systems; for example, in the affine case, we need not to restrict the dilation matrices to ones that preserve the integer lattice and are expanding on $\mathbb{R}^n$. Another novelty is a detailed analysis, in the case of affine and quasi-affine systems, of the characterizing equations for different kinds of dilation matrices.

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1 Introduction

The terms **reproducing systems** or **reproducing formulae** are applied to any of several methods that "analyze" a vector v (or function) and, then "reconstructs" v in terms of this analysis. In order to fix our ideas, let us consider a specific way in which this procedure is carried out that will help us explain the principal features of this paper.

A countable family $\{e_\alpha : \alpha \in \mathcal{A}\}$ of elements in a separable Hilbert space $\mathcal{H}$ is a **frame** if there exist constants $0 < A \le B < \infty$ satisfying

$$A\|v\|^2 \le \sum_{\alpha\in\mathcal{A}} |\langle v, e_\alpha\rangle|^2 \le B\|v\|^2$$

for all $v \in \mathcal{H}$. If only the right hand side inequality holds, we say that $\{e_\alpha : \alpha \in \mathcal{A}\}$ is a **Bessel system** with constant B. A frame is a **tight frame** if A and B can be chosen so that $A = B$, and is a **normalized tight frame** if $A = B = 1$. Thus, if $\{e_\alpha : \alpha \in \mathcal{A}\}$ is a normalized tight frame in $\mathcal{H}$, then

$$\|v\|^2 = \sum_{\alpha\in\mathcal{A}} |\langle v, e_\alpha\rangle|^2 \tag{1.1}$$

for each $v \in \mathcal{H}$. This is equivalent to

$$v = \sum_{\alpha\in\mathcal{A}} \langle v, e_\alpha\rangle\, e_\alpha \tag{1.2}$$

for all $v \in \mathcal{H}$, where the series in (1.2) converges in the norm of $\mathcal{H}$ (we refer the reader to [18], Chapters 7 and 8, for the basic properties of frames that we shall use). We shall also consider **dual systems** $\{e_\alpha : \alpha \in \mathcal{A}\}$, $\{f_\alpha : \alpha \in \mathcal{A}\}$, where the first system is used for analyzing v and the second for reconstructing v. In this case the reproducing formula has the form

$$v = \sum_{\alpha\in\mathcal{A}} \langle v, e_\alpha\rangle\, f_\alpha\,, \tag{1.3}$$

which is clearly more general than (1.2).

For the moment, in order to explain the scope of this paper, let us restrict ourselves to the case of normalized tight frames. Examples of systems that we intend to examine are the **Gabor systems**, which have the form

$$\mathcal{G}_{B,C}(g) = \{e^{2\pi i Bm\cdot x} g(x - Ck) : m, k \in \mathbb{Z}^n\}\,, \tag{1.4}$$

where $g \in L^2(\mathbb{R}^n)$ and $B, C \in GL_n(\mathbb{R})$. Another class of examples is given by the **affine systems**

$$\mathcal{F}_A(\psi) = \{\psi_{j,k}(x) = |\det A|^{j/2}\psi(A^j x - k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}, \tag{1.5}$$

where $\psi \in L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$. There are relatively simple characterizations of those functions g and ψ for which these systems (for appropriate A, B and C) are normalized tight frames for $L^2(\mathbb{R}^n)$. It is fair to say that, though the adjective "simple" is appropriate for describing the characterizations, it is not at all appropriate for a description of the proofs found in the literature (see [18, 15, 17, 29, 6, 2, 3, 8, 7]). The characterizations of those g that generate a Gabor system that is a normalized tight frame can be given by a system of equalities, and the same is true for those ψ generating affine systems that are normalized tight frames. Though these equalities are different, there are certain similarities that makes it plausible to ask if there exists a general result that contains these two characterizations as special cases. This is one of the novelties of this paper: we formulate and prove such a result (Theorem 2.1, below). Another new feature is the method of proof. It relies on an idea that appears in [19] and [23] that converts the expression on the right of equality (1.1) into a function of $x \in \mathbb{R}^n$ (here $\mathcal{H} = L^2(\mathbb{R}^n)$) by applying to v translations that depend on x; this function can then be written as an (almost periodic) Fourier series. Finally, we obtain the characterization result as a consequence of the uniqueness property for this (almost periodic) Fourier series. By these means, we obtain results that are more general than those that appear in the literature.

Perhaps, as an illustration of the type of characterization equations we are considering, it is useful to consider the affine systems (1.5) generated by a function $\psi \in L^2(\mathbb{R}^n)$. If they are a normalized tight frame, then ψ is called a normalized **tight frame wavelet** (TFW); if, in addition, $\|\psi\|_2 = 1$, the system is an orthonormal basis for $L^2(\mathbb{R}^n)$ and ψ is called an **orthonormal wavelet** or, simply, a **wavelet**. The first characterization results for such systems were obtained independently by G. Gripenberg ([16]) and X. Wang ([35]) in one dimension, and the dilation A was, simply, multiplication by 2:

Theorem 1.1. (G. Gripenberg ([16]), X. Wang ([35])) *A function $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet if and only if $\|\psi\|_2 = 1$,*

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = 1 \qquad \text{for a. e. } \xi \in \mathbb{R}, \tag{1.6}$$

and

$$t_q(\xi) = \sum_{j \geq 0} \hat{\psi}(2^j \xi) \overline{\hat{\psi}(2^j(\xi + q))} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R}, \tag{1.7}$$

whenever q is an odd integer.

Remarks. **1**. In this paper, the form of the Fourier transform we use is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi \cdot x}\, dx\,.$$

2. Without the condition $\|\psi\|_2 = 1$, the two equalities (1.6) and (1.7) characterize the normalized tight frame wavelets (as explained in [18, Chapter 7]).

Many extensions of this result were obtained in higher dimensions: for $A = 2I$ this was done in [15] and more general dilation matrices A were introduced in the references we presented after equality (1.5). Many of the proofs involve the theory of shift invariant spaces and, as a consequence, this limits the dilations A to be matrices that preserve the integer lattice $\mathbb{Z}^n$. Another assumption about A that is made in these articles is that A is **expanding** (i.e. each proper value λ satisfies

$|\lambda| > 1$). As we shall see later on, we will only need a somewhat more general hypothesis for A and do not assume that the lattice $\mathbb{Z}^n$ is preserved by A. We thus obtain a result that is more general than the characterization in [7], in which A did not have to preserve the integer lattice, but had to be expanding. In addition, we present an analysis of how the characterizing equations depend on the dilation matrix A.

The second author of this article wrote a paper ([21]) that focuses on the "unified approach" we have just described. The methods of proof in his article were based on the ideas from shift invariant spaces we mentioned above; consequently, the results obtained are less general because of the more restrictive assumptions we described in the last paragraph. The new approach also presents a good perspective of the history of the subject. For these reasons we chose the same title for this paper as the one used in [21] and added "II "at the end.

We end this introduction by indicating that the general result, Theorem 2.1, includes and leads to several applications that are more general than the ones we described above. For example, the **Gabor** and **affine** systems can be generated by finite families $\{g^1, \ldots, g^L\}$ and $\{\psi^1, \ldots, \psi^L\}$ of functions in $L^2(\mathbb{R}^n)$. Moreover, special cases involve yet other systems generated by the translation, modulations and dilations. These features are best described when we present the various applications of Theorem 2.1.

2 The main result

Let $\mathcal{P}$ be a countable collection of indices, $\{g_p : p \in \mathcal{P}\}$ be a family of functions in $L^2(\mathbb{R}^n)$ and $\{C_p : p \in \mathcal{P}\}$ be a corresponding collection of matrices in $GL_n(\mathbb{R})$. For $y \in \mathbb{R}^n$, let T_y be the translation (by y) operator defined by $T_y f = f(\cdot - y)$. The main result of this paper presents a characterization of all those families of the form

$$\left\{T_{C_p k}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\right\}, \tag{2.1}$$

that are normalized tight frames for $L^2(\mathbb{R}^n)$. We introduce the following notation:

$$\Lambda = \bigcup_{p \in \mathcal{P}} C_p^I\left(\mathbb{Z}^n\right), \tag{2.2}$$

where $C_p^I = (C_p^t)^{-1}$ (= the inverse of the transpose of C_p), and for $\alpha \in \Lambda$,

$$\mathcal{P}_\alpha = \{p \in \mathcal{P} : C_p^t\, \alpha \in \mathbb{Z}^n\}. \tag{2.3}$$

If $\alpha = 0 \in \Lambda$, then $\mathcal{P}_0 = \mathcal{P}$ (since $C_p^t\, 0 = 0$ for all $p \in \mathcal{P}$); otherwise the best we can say is that $\mathcal{P}_\alpha \subset \mathcal{P}$.

Let N be defined on $L^2(\mathbb{R}^n)$ by letting

$$N^2(f) = \sum_{p \in \mathcal{P}} \sum_{k \in \mathbb{Z}^n} |\langle f, T_{C_p k}\, g_p\rangle|^2 \tag{2.4}$$

for $f \in L^2(\mathbb{R}^n)$. By (1.1), the system (2.1) is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if N is the $L^2(\mathbb{R}^n)$-norm of f:

$$N^2(f) = \|f\|_2^2 \tag{2.5}$$

for all $f \in L^2(\mathbb{R}^n)$. Our main result, therefore, involves conditions on the system (2.1) that are equivalent to equality (2.5).

Since equalities (1.2) and (1.1) are valid for all $v \in \mathcal{H}$ if and only if they hold for a dense subspace of $\mathcal{H}$ (see [18, Chapter 7]), we will find it useful to introduce the set

$$\mathcal{D} = \{f \in L^2(\mathbb{R}^n) \colon \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp}\hat{f} \text{ is compact}\},$$

which is dense in $L^2(\mathbb{R}^n)$.

Here is the statement of our main result:

Theorem 2.1. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$ and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. Suppose that*

$$L(f) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\,\hat{f}} |\hat{f}(\xi + C_p^I m)|^2 \frac{1}{|\det C_p|} |\hat{g}_p(\xi)|^2 \, d\xi < \infty \tag{2.6}$$

for all $f \in \mathcal{D}$, where $C_p^I = (C_p^t)^{-1}$. Then the system (2.1) is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if

$$\sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|} \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi+\alpha) = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{2.7}$$

for each $\alpha \in \Lambda$, where δ is the Kronecker delta for $\mathbb{R}^n$.

The proof of this result will be derived from some lemmas that will be established in this section. In the course of doing so we shall also indicate why the hypothesis (2.6) is plausible and discuss the convergence of some of the series we shall encounter. As a first observation along these lines, note that if equality (2.7) is valid for $\alpha = 0$, so that

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|} |\hat{g}_p(\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{2.8}$$

then if follows from Schwarz's inequality that the other series in (2.7) are a.e. absolutely convergent (recall that $\mathcal{P}_\alpha \subset \mathcal{P}$).

Let C be an $n\times n$ real matrix and $f, g \in L^2(\mathbb{R}^n)$. The **C-bracket product** of f and g is defined as

$$[f,g](x;C) \;=\; \sum_{k\in\mathbb{Z}^n} f(x-Ck)\,\overline{g(x-Ck)}. \tag{2.9}$$

This is an extension of the notion and notation introduced in [11] when $C = I$. It is clear that $[f,g]$ is $C\mathbb{Z}^n$- periodic; that is, $[f,g](x+Cm;C) = [f,g](x;C)$ for each $m \in \mathbb{Z}^n$.

Lemma 2.2. *Let $C \in GL_n(\mathbb{R})$ and $C^I = (C^t)^{-1}$. If $f \in \mathcal{D}$ and $g \in L^2(\mathbb{R}^n)$, then*

$$\sum_{k\in\mathbb{Z}^n} |\langle f, T_{Ck}\, g\rangle|^2 = \frac{1}{|\det C|} \int_{C^I\mathbb{T}^n} |[\hat{f}, \hat{g}](\xi; C^I)|^2 \, d\,\xi, \tag{2.10}$$

where $\mathbb{T}^n = [0,1)^n$.

Proof. Since $(T_{Ck}\, g)^\wedge(\xi) = e^{-2\pi i Ck\cdot\xi}\,\hat{g}(\xi)$, it follows from the Plancherel theorem that the left side of (2.10) equals

$$\sum_{k\in\mathbb{Z}^n} \Big| \int_{\mathbb{R}^n} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\, e^{2\pi i Ck\cdot\xi}\, d\xi \Big|^2 . \tag{2.11}$$

Since $\mathbb{R}^n = \bigcup_{l\in\mathbb{Z}^n} \{C^I(\mathbb{T}^n - l)\}$ is a disjoint union, the integral in (2.11) can be written in the form

$$\sum_{l\in\mathbb{Z}^n} \int_{C^I(\mathbb{T}^n)} \hat{f}(\xi - C^I l)\,\overline{\hat{g}(\xi - C^I l)}\, e^{2\pi i Ck\cdot\xi}\, d\xi = \int_{C^I(\mathbb{T}^n)} [\hat{f}, \hat{g}](\xi; C^I)\, e^{2\pi i Ck\cdot\xi}\, d\xi \,.$$

But $[\hat{f}, \hat{g}](\xi; C^I)$ is a $C^I\mathbb{Z}^n$-periodic function belonging to $L^2(C^I\mathbb{T}^n)$ (since $f \in \mathcal{D}$). Thus, the expression (2.11) is, up to a constant, the square of the l^2-norm of the Fourier coefficients of this $C^I\mathbb{Z}^n$-periodic function with respect to the orthonormal basis

$$\{\sqrt{|\det C|}\, e^{2\pi i Ck\cdot\xi} : k \in \mathbb{Z}^n\}$$

of $L^2(C^I\mathbb{T}^n)$. Equality (2.10) now follows immediately from this observation. □

Lemma 2.3. *Let $C \in GL_n(\mathbb{R})$ and $C^I = (C^t)^{-1}$. For each $f \in \mathcal{D}$ and $g \in L^2(\mathbb{R}^n)$, the function*

$$H(x) \;=\; \sum_{k\in\mathbb{Z}^n} |\langle T_x\, f, T_{Ck}\, g\rangle|^2 \tag{2.12}$$

is the trigonometric polynomial

$$H(x) = \sum_{m\in\mathbb{Z}^n} \hat{H}(m)\, e^{2\pi i (C^I m)\cdot x},$$

where

$$\hat{H}(m) \;=\; \frac{1}{|\det C|} \int_{\mathbb{R}^n} \hat{f}(\xi)\,\overline{\hat{f}(\xi + C^I m)}\,\overline{\hat{g}(\xi)}\,\hat{g}(\xi + C^I m)\, d\xi, \tag{2.13}$$

and only a finite number of these expressions is non-zero.

Proof. If we do establish (2.13), the fact that $\hat{H}(m) = 0$ for all but finitely many m is an immediate consequence of the fact that $\hat{f}(\xi)$ and $\hat{f}(\xi + C^I m)$ must have disjoint support if $|m|$ is sufficiently large. By Lemma 2.2,

$$\begin{aligned}
|\det C|\, H(x) &= \int_{C^I\mathbb{T}^n} |[(T_x\, f)^\wedge, \hat{g}](\xi; C^I)|^2 \, d\,\xi \\
&= \int_{C^I\mathbb{T}^n} \Big| e^{-2\pi i \xi\cdot x} \sum_{m\in\mathbb{Z}^n} e^{-2\pi i C^I m\cdot x} \hat{f}(\xi + C^I m)\,\overline{\hat{g}(\xi + C^I m)} \Big|^2 \, d\xi \\
&= \int_{C^I\mathbb{T}^n} \sum_{m\in\mathbb{Z}^n} e^{-2\pi i C^I m\cdot x} \hat{f}(\xi + C^I m)\,\overline{\hat{g}(\xi + C^I m)} \sum_{l\in\mathbb{Z}^n} e^{2\pi i C^I l\cdot x}\,\overline{\hat{f}(\xi + C^I l)}\,\hat{g}(\xi + C^I l)\, d\xi \,.
\end{aligned}$$

Let $k = l - m$ and express the above integrand function as a sum over k and m. We obtain the expression

$$\sum_{m\in\mathbb{Z}^n} \int_{C^I\mathbb{T}^n} \hat{f}(\xi + C^I m)\, \overline{\hat{g}(\xi + C^I m)} \sum_{k\in\mathbb{Z}^n} e^{2\pi i C^I k\cdot x}\, \overline{\hat{f}(\xi + C^I m + C^I k)}\, \hat{g}(\xi + C^I m + C^I k)\; d\xi$$

$$= \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{g}(\xi)} \sum_{k\in\mathbb{Z}^n} e^{2\pi i C^I k\cdot x}\, \overline{\hat{f}(\xi + C^I k)}\, \hat{g}(\xi + C^I k)\; d\xi$$

$$= \sum_{k\in\mathbb{Z}^n} \Big(\int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C^I k)}\, \overline{\hat{g}(\xi)}\, \hat{g}(\xi + C^I k)\, d\xi \Big)\, e^{2\pi i C^I k\cdot x}.$$

The various exchanges of summations and integration are justified by the fact that $f \in \mathcal{D}$. Equality (2.13) is obtained by dividing by $|\det C|$. $\square$

We are now ready to state and prove the principal result that we shall use to establish Theorem 2.1:

Proposition 2.4. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$, $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$, and let $C_p^I = (C_p^t)^{-1}$. Assume that, for $f \in \mathcal{D}$, (2.6) is valid.*

$$\sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\,\hat{f}} |\hat{f}(\xi + C_p^I m)|^2\, \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2\, d\xi < \infty. \tag{2.14}$$

Then, the function

$$w(x) = N^2(T_x\, f) \;=\; \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle T_x f, T_{C_p k}\, g_p\rangle|^2$$

is a continuous function that coincides pointwise with its absolutely convergent (almost periodic) Fourier series

$$\sum_{\alpha\in\Lambda} \hat{w}(\alpha)\, e^{2\pi i\alpha\cdot x}\ ,$$

where

$$\hat{w}(\alpha) \;=\; \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + \alpha)} \sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + \alpha)\, d\xi, \tag{2.15}$$

and the integral in (2.15) converges absolutely.

Remark. The function $w(x)$ given in the above proposition is an almost periodic function since these are characterized as uniform limits of generalized trigonometric polynomials (see [1]).

Proof. Observe that

$$w(x) = N^2(T_x\, f) = \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle f, T_{C_p(k - C_p^{-1}x)}\, g_p\rangle|^2.$$

For a fixed $p \in \mathcal{P}$, let $w_p(x)$ denote the above sum over $k \in \mathbb{Z}^n$. By Lemma 2.3, $w_p(x)$ is the $C_p\mathbb{Z}^n$-periodic trigonometric polynomial

$$w_p(x) = \sum_{m\in\mathbb{Z}^n} \hat{w}_p(m)\, e^{2\pi i C_p^I m\cdot x},$$

where

$$\hat{w}_p(m) = \frac{1}{|\det C_p|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C_p^I m)}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I m)\, d\xi. \tag{2.16}$$

We claim that $\{\hat{w}_p(m) : p \in \mathcal{P}, m \in \mathbb{Z}^n\}$ belongs to $\ell^1(\mathcal{P} \times \mathbb{Z}^n)$. To see this, let $K = \operatorname{supp} \hat{f}$ (recall that $f \in \mathcal{D}$ and, thus, K is compact) and $K(m) = K - C_p^I\, m$, so that $\hat{f}(\xi)\overline{\hat{f}(\xi + C_p^I\, m)} \neq 0$ only if $\xi \in K \cap K(m)$. Thus, the integral over $\mathbb{R}^n$ in (2.16) is really over this intersection. An application of Schwarz's inequality then gives us the fact that this integral does not exceed

$$\left(\int_{K(m)} |\hat{f}(\xi)\, \hat{g}_p(\xi + C_p^I m)|^2\, d\xi \right)^{1/2} \left(\int_K |\hat{f}(\xi + C_p^I m)\, \hat{g}_p(\xi)|^2\, d\xi \right)^{1/2}$$

and the change of variables $\xi = \eta - C_p^I m$ in the first integral makes this expression equal to

$$\left(\int_K |\hat{f}(\eta - C_p^I m)\, \hat{g}_p(\eta)|^2\, d\eta \right)^{1/2} \left(\int_K |\hat{f}(\xi + C_p^I m)\, \hat{g}_p(\xi)|^2\, d\xi \right)^{1/2}.$$

Then the inequality $2|cd| \leq |c|^2 + |d|^2$ together with condition (2.6) proves

$$\sum_{p \in \mathcal{P}} \sum_{m \in \mathbb{Z}^n} |\hat{w}_p(m)| < \infty,$$

which is our claim. It follows that

$$w(x) = \sum_{p \in \mathcal{P}} \sum_{m \in \mathbb{Z}^n} \hat{w}_p(m)\, e^{2\pi i C_p^I m \cdot x}$$

where the convergence is absolute and uniform. In terms of the notation introduced in (2.2) and (2.3), we can write this last equality in the form

$$\begin{aligned} w(x) &= \sum_{\alpha \in \Lambda} \Big\{ \sum_{p \in \mathcal{P}_\alpha} \int_{\mathbb{R}^n} \frac{1}{|\det C_p|}\, \hat{f}(\xi)\, \overline{\hat{f}(\xi + \alpha)}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + \alpha)\, d\xi \Big\}\, e^{2\pi i \alpha \cdot x} \\ &= \sum_{\alpha \in \Lambda} \Big\{ \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + \alpha)} \sum_{p \in \mathcal{P}_\alpha} \frac{1}{|\det C_p|}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + \alpha)\, d\xi \Big\}\, e^{2\pi i \alpha \cdot x} \\ &= \sum_{\alpha \in \Lambda} \hat{w}(\alpha)\, e^{2\pi i \alpha \cdot x}, \end{aligned} \tag{2.17}$$

where $\hat{w}(\alpha)$ is the sum of some of the coefficients $\hat{w}_p(m)$, as indicated within the curly bracket. Since, as we have shown, $\{\hat{w}_p(m) : p \in \mathcal{P}, m \in \mathbb{Z}^n\}$ belongs to $\ell^1(\mathcal{P} \times \mathbb{Z}^n)$, it follows that $\{\hat{w}(\alpha) : \alpha \in \Lambda\}$ belongs to $\ell^1(\Lambda)$. Then it immediately follows that the last series in (2.17) is absolutely convergent. This finishes the proof of the proposition. □

Remark. Notice that condition (2.6) has been used to prove that $\{\hat{w}(\alpha) : \alpha \in \Lambda\}$ belongs to $\ell^1(\Lambda)$. As we shall see, in most cases, when we apply Theorem 2.1 we do not need to assume condition (2.6); for example, it will be shown that the Gabor systems, the affine systems and some related systems do satisfy this property.

The following lemma, that will be needed in the proof of Theorem 2.1, is a simple fact about uniqueness of the coefficients of an almost periodic Fourier series, as the one in (2.17).

Lemma 2.5. *Suppose* $\{c_\alpha : \alpha \in \Lambda\} \in \ell^1(\Lambda)$ *where* $\Lambda \subset \mathbb{R}^n$ *is countable. Then,*

$$v(x) = \textstyle\sum_{\alpha\in\Lambda} c_\alpha\, e^{2\pi i\alpha\cdot x} = 0 \textit{ for all } x \in \mathbb{R}^n \textit{ if and only if } c_\alpha = 0 \textit{ for all } \alpha \in \Lambda.$$

Proof. It is clear that if $c_\alpha = 0$ for all $\alpha \in \Lambda$, then $v(x) \equiv 0$. Suppose $v(x) = 0$ for all $x \in \mathbb{R}^n$. Fix $\beta \in \Lambda$ and let $Q(R) = [-R, R]^n$, $R > 0$. Then

$$0 = \lim_{R\to\infty} \frac{1}{(2R)^n} \int_{Q(R)} v(x)\, e^{-2\pi i\beta\cdot x}\, dx = \lim_{R\to\infty} \sum_{\alpha\in\Lambda} c_\alpha \frac{1}{(2R)^n} \int_{Q(R)} e^{2\pi i\alpha\cdot x}\, e^{-2\pi i\beta\cdot x}\, dx\,.$$

Let us examine each of the above integral means. If $\alpha = \beta$, then the mean is 1. If $\alpha \neq \beta$, then

$$\frac{1}{(2R)^n} \int_{Q(R)} e^{2\pi i(\alpha-\beta)\cdot x}\, dx = \prod_{j=1}^{n} \Big\{ \frac{1}{2R} \int_{-R}^{R} e^{-2\pi i(\alpha j-\beta_j)x}\, dx \Big\}\,.$$

For at least one j, $\alpha_j - \beta_j \neq 0$. Thus, this factor is equal to

$$\frac{1}{2R} \frac{2\sin(2\pi(\alpha_j - \beta_j)R)}{2\pi(\alpha_j - \beta_j)}\,,$$

which tends to zero as $R \to \infty$. $\square$

Proof of Theorem 2.1. As observed before the statement of Theorem 2.1, it suffices to prove the result for a dense subset of $L^2(\mathbb{R}^n)$. Let us assume that condition (2.6) holds for all $f \in \mathcal{D}$ and that (2.7) is true. By Proposition 2.4,

$$w(x) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} |\langle T_x\, f, T_{C_p m}\, g_p\rangle|^2 = \sum_{\alpha\in\Lambda} \hat{w}(\alpha)\, e^{2\pi i\alpha\cdot x},$$

where the last series converges absolutely (thus, $w(x)$ is continuous) and, by (2.7) and (2.15),

$$\hat{w}(\alpha) \;=\; \Big(\int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi+\alpha)}\, d\xi \Big)\, \delta_{\alpha,0}$$

for each $f \in \mathcal{D}$. The desired tight frame property (2.5) follows by letting $x = 0$. Now let us assume that we have the tight frame property $N^2(f) = \|f\|^2$ for all $f \in L^2(\mathbb{R}^n)$. By Proposition 2.4, if $f \in \mathcal{D}$, then the function $z(x) = w(x) - \|f\|^2$ is continuous and equals an absolutely convergent (generalized) trigonometric series whose coefficients are

$$\hat{z}(0) = \hat{w}(0) - \|f\|^2, \qquad \text{and} \qquad \hat{z}(\alpha) = \hat{w}(\alpha),\ \alpha \neq 0.$$

Since $z(x) = 0$, it follows from Lemma 2.5 that all coefficients $\hat{z}(\alpha)$ must be 0. Thus, for $\alpha \in \Lambda$ and $f \in \mathcal{D}$, we have

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi+\alpha)} \Big(\sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi+\alpha) \Big)\, d\xi = \delta_{\alpha,0}\, \|f\|^2. \tag{2.18}$$

Consider the case $\alpha = 0$ and let

$$h_0(\xi) = \sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2\,.$$

By (2.6), h_0 is locally integrable; choose ξ_0 to be a point of differentiability of the integral of this function. Letting $B(\epsilon)$ denote the ball of radius $\epsilon > 0$ about the origin, define f_ϵ by

$$\hat{f}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}} \chi_{B(\epsilon)}(\xi - \xi_0) \, .$$

Then $\|f_\epsilon\|_2 = 1$ and $f_\epsilon \in \mathcal{D}$. By (2.18) with $f = f_\epsilon$, we have

$$1 = \lim_{\epsilon \to 0} \int_{|\xi - \xi_0| \le \epsilon} \frac{1}{|B(\epsilon)|} h_0(\xi) \, d\xi = h_0(\xi_0).$$

This shows that $h_0(\xi) = 1$, a.e. $\xi \in \mathbb{R}^n$, and (2.7) is satisfied for $\alpha = 0$. When $\alpha \neq 0$, let

$$h_\alpha(\xi) = \sum_{p \in \mathcal{P}(\alpha)} \frac{1}{|\det C_p|} \overline{\hat{g}_p(\xi)} \, \hat{g}_p(\xi + \alpha).$$

By polarization of (2.18) we have

$$\int_{\mathbb{R}^n} \hat{f}(\xi) \, \overline{\hat{\phi}(\xi + \alpha)} \, h_\alpha(\xi) \, d\xi = 0 \tag{2.19}$$

for all $f, \phi \in \mathcal{D}$. By Schwarz's inequality and (2.6), h_α is locally integrable. We can choose, again, a point of differentiability ξ_0 of the integral of h_α, and choose f_ϵ and ϕ_ϵ such that

$$\hat{f}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}} \chi_{B(\epsilon)}(\xi - \xi_0), \qquad \hat{\phi}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}} \chi_{B(\epsilon)}(\xi - \xi_0 - \alpha).$$

Hence $\|f_\epsilon\|_2 = \|\phi_\epsilon\|_2 = 1$, f_ϵ , $\phi_\epsilon \in D$ and, by (2.19),

$$0 = \lim_{\epsilon \to 0} \int_{|\xi - \xi_0| \le \epsilon} \frac{1}{|B(\epsilon)|} h_\alpha(\xi) \, d\xi = h_\alpha(\xi_0).$$

Hence $h_\alpha(\xi) = 0$, a.e. $\xi \in \mathbb{R}^n$, and (2.7) is satisfied for $\alpha \neq 0$. □

Remark. In some applications, namely in the case of affine systems, it will be useful to replace the dense set $\mathcal{D}$ that appears in the statement of Theorem 2.1 by smaller dense sets of the form

$$\mathcal{D}_E = \big\{ f \in L^2(\mathbb{R}^n) \colon \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp} \hat{f} \text{ is compact in } \mathbb{R}^n \setminus E \big\},$$

for any linear subspace E of $\mathbb{R}^n$ of dimension smaller than n. The result of Theorem 2.1 still holds true if the set $\mathcal{D}$ is replaced by any of these smaller dense sets.

3 The Gabor systems

Given a function $g \in L^2(\mathbb{R})$ and $b, c \in \mathbb{R} \setminus \{0\}$, then the classical **Gabor system** on $\mathbb{R}$ generated by g with parameters b and c is the collection

$$\mathcal{G}_{b,c}(g) = \{ e^{2\pi i b m x} g(x - ck) : m, k \in \mathbb{Z} \} \, . \tag{3.1}$$

Many results are known that determine conditions on g and relations between the parameters for such systems to be a frame (see, for example, [18], where the Balian-Low theorem is presented, the density theorem of Rieffel ([27, 22, 32]) and the duality condition ([19, 12, 30])). We begin by showing that Theorem 2.1 can be applied directly for obtaining a characterization of those n-dimensional extensions of the system (3.1) that are normalized tight frames. The results we obtain include characterizations obtained by different authors ([30, 10, 21]). In order to describe these systems we will use the **translation operators** (as defined in Section 2) and the **modulation operators** M_z, $z \in \mathbb{R}^n$ defined by

$$(M_z\, f)(x) = e^{2\pi i z\cdot x}\, f(x),$$

for $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The Gabor systems will be generated by a finite family $G = \{g^1, g^2, \dots, g^L\} \subset L^2(\mathbb{R}^n)$ and a pair of matrices $B, C \in GL_n(\mathbb{R})$ so that they have the form

$$\mathcal{G} = \mathcal{G}_{B,C}(G) = \{M_{Bm} T_{Ck}\, g^\ell : m, k \in \mathbb{Z}^n,\ \ell = 1, 2, \cdots, L\}\,. \tag{3.2}$$

If we change the order in which the translation and modulation operators are applied we also have the system

$$\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_{B,C}(G) = \{T_{Ck} M_{Bm}\, g^\ell : m, k \in \mathbb{Z}^n,\ \ell = 1, 2, \cdots, L\}\,. \tag{3.3}$$

A simple calculation shows that

$$T_{Ck} M_{Bm}\, g^\ell = e^{-2\pi i Bm\cdot Ck} M_{Bm} T_{Ck}\, g^\ell \tag{3.4}$$

$m, k \in \mathbb{Z}^n$, and it follows immediately that

Lemma 3.1. *(***a***) $\mathcal{G}$ is a frame for $L^2(\mathbb{R}^n)$ if and only if $\widetilde{\mathcal{G}}$ is a frame for $L^2(\mathbb{R}^n)$; furthermore, the frame constants A and B can be taken to be the same in the two cases. (***b***) $\mathcal{G}$ is an orthonormal system if and only if $\widetilde{\mathcal{G}}$ is an orthonormal system.*

We begin by observing that our main result, Theorem 2.1, easily implies the following characterization theorem:

Theorem 3.2. *The system $\mathcal{G} = \mathcal{G}_{B,C}(G)$ (or $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_{B,C}(G)$) is a normalized tight frame if and only if*

$$\sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} \frac{1}{|\det C|}\, \hat{g}^\ell(\xi - Bk)\, \overline{\hat{g}^\ell(\xi - Bk + C^I m)} = \delta_{m,0} \tag{3.5}$$

for a.e. $\xi \in \mathbb{R}^n$, all $m \in \mathbb{Z}^n$, where $C^I = (C^t)^{-1}$.

Proof. It will be clear from our proof that we can reduce the argument by assuming $L = 1$; in any case, we shall address this issue after we show how to apply Theorem 2.1. By Lemma 3.1, it suffices to consider the system $\widetilde{\mathcal{G}}$. When we do this, we can write it in the form (2.1) by letting $g_p = M_{Bp}\, g$ for $p \in \mathcal{P} = \mathbb{Z}^n$ and $C_p = C$. Condition (2.6) follows: for $f \in \mathcal{D}$, only a finite number of terms of the form $\hat{f}(\xi + C^I\, m)$ can be non-zero if ξ is restricted to $K = \operatorname{supp} \hat{f}$ (recall that C and, therefore, C^I, are invertible and that K is bounded). Hence the integrability over K of

$$\sum_{p\in\mathbb{Z}^n} \frac{1}{|\det(C)|} \sum_{m\in\mathbb{Z}^n} |\hat{f}(\xi + C^I\, m)\, \hat{g}(\xi - B\, p)|^2$$

follows from the integrability over K of

$$\sum_{p\in\mathbb{Z}^n} |\hat{f}(\xi + C^I\, m)\, \hat{g}(\xi - B\, p)|^2$$

for each $m \in \mathbb{Z}^n$ (since all but a finite number of these expressions is non-zero; also recall that $\mathcal{P} = \mathbb{Z}^n$ in our present case). Furthermore, the fact that $\|\hat{f}\|_\infty < \infty$ reduces our task to showing that

$$\int_K \sum_{p\in\mathbb{Z}^n} |\hat{g}(\xi - Bp)|^2 < \infty\,. \tag{3.6}$$

For each $j \in \mathbb{Z}^n$, the collection $\{B(\mathbb{T}^n + j - p) : p \in \mathbb{Z}^n\}$ is a partition of $\mathbb{R}^n$. Thus,

$$\|g\|_2^2 = \int_{\bigcup_{p\in\mathbb{Z}^n} B(\mathbb{T}^n+j-p)} |\hat{g}(\eta)|^2\, d\eta = \int_{B(\mathbb{T}^n+j)} \sum_{p\in\mathbb{Z}^n} |\hat{g}(\xi - B\, p)|^2\, d\xi$$

which shows the integrability of the integrand in (3.6) over the set $B(\mathbb{T}^n + j)$ for each $j \in \mathbb{Z}^n$. Since any bounded subset of $\mathbb{R}^n$ is contained in a finite number of such sets, we have the desired integrability. Incidentally, if we had $L > 1$, this proves the local integrability of L sums of the form (3.6). Theorem 3.2 now follows from Theorem 2.1 using $(M_{Bp}\, g)^\wedge(\xi) = \hat{g}(\xi - Bp)$. □

4 The Calderón condition and reproducing systems

As mentioned after the statement of Theorem 2.1, the case $\alpha = 0$ of (2.7) is

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{4.1}$$

This formula is valid when the systems described in (2.1) are normalized tight frames and satisfy condition (2.6). When applying this result to the affine system (1.5), a simple calculation (see also Section 6) shows that (4.1) becomes

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^j\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{4.2}$$

for $\psi \in L^2(\mathbb{R}^n)$ and $B = A^t \in GL_n(\mathbb{R})$. Versions of the "resolution of the identity"(4.2) have appeared in works of A. P. Calderón and has become known as the Calderón condition in the area of orthonormal wavelets. For this reason we shall say that (4.1) is a **Calderón condition**.

Under the assumption (2.6), Theorem 2.1 shows that the Calderón condition (4.1) is necessary for the system $\{T_{C_p k}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$, given by(2.1), to be a normalized tight frame. Together with the cases $\alpha \neq 0$ of (2.7) we obtain a necessary and sufficient condition. We will show in this section that other type of conditions can replace the cases $\alpha \neq 0$ of (2.7).

If we remove condition (2.6) we can still prove a weaker version of(4.1) where the equality is replaced by an inequality. This result,which will play a major role in Section 5, is a consequence of Lemma 2.3 and it is given below. The result is stated and proved for Bessel systems as defined in Section 1.

Proposition 4.1. *Let $\mathcal{P}$ be a countable set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$, and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. If the system $\{T_{C_p k}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is Bessel with constant B, then*

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2 \le B \qquad \textit{for a.e. } \xi \in \mathbb{R}^n. \tag{4.3}$$

Proof. In most applications of this Proposition, $\mathcal{P}$ will be a subset of $\mathbb{Z}^r$ for some $r \in \mathbb{N}$. For simplicity we assume this to be the case here. However, the reader can easily check that this is not a loss of generality.

Assume that $\{T_{C_p k}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a Bessel sequence with constant B. Then, for every $M \in \mathbb{N}$

$$\sum_{p\in\mathcal{P},|p|\le M} \sum_{k\in\mathbb{Z}^n} |\langle f, T_{C_p k}\, g_p\rangle|^2 \le B\|f\|_2^2$$

for all $f \in L^2(\mathbb{R}^n)$. Applying Lemmas 2.2 and 2.3 to each $p \in \mathcal{P}$ (letting $x = 0$), we can write

$$\sum_{p\in\mathcal{P},|p|\le M} \sum_{k\in\mathbb{Z}^n} \frac{1}{|\det C_p|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C_p^I\, k)}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I\, k)\, d\xi \le B\|f\|_2^2 \tag{4.4}$$

for all $f \in \mathcal{D}$, $M \in \mathbb{N}$ (also recall that $C^I = (C^t)^{-1}$). For each $M \in \mathbb{N}$ let

$$h_{0,M}(\xi) = \sum_{p\in\mathcal{P},|p|\le M} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2\,.$$

Since each $g_p \in L^2(\mathbb{R}^n)$ and there is only a finite number of elements of $\mathcal{P}$ in the above sum, $h_{0,M} \in L^1(\mathbb{R}^n)$. Let L_M be the set of differentiability points of the integral of $h_{0,M}$ and take $\xi_0 \in L_M$. Letting $B(\epsilon)$ denote the ball of radius $\epsilon > 0$ about the origin, define f_ϵ by

$$\hat{f}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}}\, \chi_{B(\epsilon)}(\xi - \xi_0)\,.$$

Then $\|f_\epsilon\|_2 = 1$ and $f_\epsilon \in \mathcal{D}$. For $M \in \mathbb{N}$, let

$$\Lambda_{0,M} = \bigcup_{p\in\mathcal{P},|p|\le M} C_p^I(\mathbb{Z}^n \setminus \{0\}) \qquad \text{and} \qquad \delta_M = \inf\{|C_p^I\, k| : C_p^I\, k \in \Lambda_{0,M}\}.$$

Observe that $\delta_M > 0$ since each C_p^I is invertible, $k \ne 0$, and there is only a finite number of elements of $\mathcal{P}$ in the set $\Lambda_{0,M}$. For $\epsilon < \delta_M/2$, $|\xi - \xi_0| < \epsilon$, and $C_p^I\, k \in \Lambda_{0,M}$ we have

$$|\xi + C_p^I\, k - \xi_0| \ge |C_p^I\, k| - |\xi - \xi_0| \ge \delta_M - \epsilon > \delta_M - \frac{\delta_M}{2} = \frac{\delta_M}{2} > \epsilon\,,$$

so that $\xi + C_p^I\, k - \xi_0$ does not belong to $B(\epsilon)$. This means that $\hat{f}_\epsilon(\xi + C_p^I\, k) = 0$ for all $k \ne 0$, $\epsilon < \delta_M/2$, and $|p| < M$, and, thus, all the terms in (4.4) equal 0 except the one corresponding to $k = 0$. Thus,

$$\lim_{\epsilon\to 0} \int_{|\xi-\xi_0|\le\epsilon} \frac{1}{|B(\epsilon)|}\, h_{0,M}(\xi)\, d\xi = \lim_{\epsilon\to 0} \int_{|\xi-\xi_0|\le\epsilon} \frac{1}{|B(\epsilon)|} \sum_{p\in\mathcal{P},|p|\le M} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2\, d\xi \le B.$$

Since the left hand side of this formula coincides with $h_{0,M}(\xi_0)$, we deduce that $h_{0,M}(\xi_0) \le B$ for all $\xi_0 \in L_M$.

Since

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2 \;=\; \lim_{M\to\infty} h_{0,M}(\xi),$$

we obtain the desired result for all ξ in the intersection of all L_M, which is a dense set in $\mathbb{R}^n$. $\square$

We now present the main result of this section, which follows from the arguments presented in Section 2.

Theorem 4.2. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$ and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. Suppose that (2.6) holds for all $f \in \mathcal{D}$. Then the system $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if it is a Bessel system with constant 1 and the Calderón condition (4.1) holds.*

Proof. Under condition (2.6), if $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a normalized tight frame, then it is clearly Bessel with constant 1, and, by Theorem 2.1, the Calderón condition (4.1) holds (take $\alpha = 0$ in (2.7)).

For the converse we need to recall the following fact about almost periodic functions which can be found in [1, Satz XXXVI] (see also [36, page 111]):

Lemma 4.3. *Suppose that h is a non-negative almost periodic function defined in $\mathbb{R}^n$, and let*

$$M(h) \equiv \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} h(x)\, dx$$

be the mean of h, where $Q(R) = [-R, R]^n$. Then, $M(h) = 0$ if and only if $h \equiv 0$.

Let

$$w(x) = \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle T_x f, T_{C_pk}\, g_p\rangle|^2$$

as in Proposition 2.4. Since $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is Bessel with constant 1 we have $w(x) \le \|T_x f\|_2^2 = \|f\|_2^2$ for all $f \in L^2(\mathbb{R}^n)$. Thus, for any $f \in \mathcal{D}$, the function $h(x) = \|f\|_2^2 - w(x)$ is non-negative, and, by Proposition 2.4 and the remark that follows its proof, is continuous and almost periodic. Taking the mean value of $h(x)$ and using, again, Proposition 2.4 to write $w(x)$ as an absolutely convergent (generalized) Fourier series with coefficients $\hat{w}(\alpha)$, given by (2.15), we obtain

$$M(h) = \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} h(x)\, dx = \|f\|_2^2 - \sum_{\alpha\in\Lambda} \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} \hat{w}(\alpha)\, e^{2\pi i\alpha\cdot x}\, dx\,.$$

As in the proof of Lemma 2.5 all the above integrals are zero except the one corresponding to $\alpha = 0$ that becomes $\hat{w}(0)$. Thus, $M(h) = \|f\|_2^2 - \hat{w}(0)$ for all $f \in \mathcal{D}$. By the Calderón condition (4.1), we have

$$\hat{w}(0) = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2\, d\xi = \|f\|_2^2\,.$$

Hence, $M(h) = 0$ for all $f \in \mathcal{D}$. By Lemma 4.3, $h(x) = 0$ for all $x \in \mathbb{R}^n$ and all $f \in \mathcal{D}$. Taking $x = 0$ (recall that h is continuous) we deduce that $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a normalized tight frame, as desired, since $\mathcal{D}$ is dense in $L^2(\mathbb{R}^n)$. $\square$

Remarks. (**1**) That Theorem 4.2 follows from our main work in Section 2 was pointed out to us by S. Xiao. The method that we use follows the line of argument presented in [24]. In the case of wavelet systems, like the types described by (1.5), Theorem 4.2 has been proved by M. Bownik [3] for expanding dilation matrices with integer entries, and by R. Laugesen [23, 24] for expanding dilation matrices with real entries.

(**2**) As in the remark given at the end of Section 2, the set $\mathcal{D}$ that appears in Theorem 4.2 can be replaced by the smaller dense subsets $\mathcal{D}_E$ and Theorem 4.2 still holds.

For orthonormal systems, we have the following simple corollary of Theorem 4.2:

Corollary 4.4. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$ and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. Suppose that (2.6) holds for all $f \in \mathcal{D}$ and that the system $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is an orthonormal system in $L^2(\mathbb{R}^n)$. Then $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is complete in $L^2(\mathbb{R}^n)$ if and only if the Calderón condition (4.1) holds.*

When applied to wavelet systems, like the types described in (1.5), Corollary 4.4 shows that an orthonormal wavelet system is complete if and only if the Calderón condition for wavelets (4.2) holds. This has been proved in [3, 33] for expanding dilation matrices with integer entries and in [24] and for expanding dilation matrices with real entries.

In the next section, we shall explain how Theorem 4.2 and Corollary 4.4 can be applied to the affine systems. For the moment, we restrict our attention to the Gabor systems and establish other consequences of Theorem 4.2 and Corollary 4.4.

Consider the Gabor systems $\mathcal{G}_{B,C}(G)$, given by (3.2), and $\widetilde{\mathcal{G}}_{B,C}(G)$, given by (3.3). Since $(M_{B\,p}\, g)^\wedge(\xi) = \hat{g}(\xi - Bp)$, the Calderón condition (4.1) for the system $\widetilde{\mathcal{G}}_{B,C}(G)$ becomes

$$\sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} |\hat{g^\ell}(\xi - B\,k)|^2 = |\det C| \qquad \text{for a. e. } \xi \in \mathbb{R}^n. \tag{4.5}$$

¿From Theorem (4.2) and Corollary (4.4) we obtain:

Corollary 4.5. *Let $G = \{g^1, \ldots, g^L\} \subset L^2(\mathbb{R}^n)$ and $B, C \in GL_n(\mathbb{R})$. Then the Gabor system $\mathcal{G}$ (or $\widetilde{\mathcal{G}}$) is a normalized tight frame if and only if $\mathcal{G}$ (or $\widetilde{\mathcal{G}}$) is a Bessel system with constant 1 and (4.5) holds.*

Corollary 4.6. *Let $G = \{g^1, \ldots, g^L\} \subset L^2(\mathbb{R}^n)$ and $B, C \in GL_n(\mathbb{R})$. Suppose that the Gabor system $\mathcal{G}$ (or $\widetilde{\mathcal{G}}$) is an orthonormal system in $L^2(\mathbb{R}^n)$. Then $\mathcal{G}$ (or $\widetilde{\mathcal{G}}$) is complete if and only if (4.5) holds.*

The results obtained in the above corollaries are contained in [30, 20, 10]. Thus, neither of these two results are new, but the point is that each follows easily from our general framework.

Using Proposition 4.1, we obtain the following special case of Theorem 2.1, where we assume $C_p = C$ for every $p \in \mathcal{P}$. This result can also be found in [28, 21]. Observe that, as it is clear from the proof, we do not need condition (2.6)

Theorem 4.7. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}$ be a collection of functions in $L^2(\mathbb{R}^n)$ and $C \in GL_n(\mathbb{R})$. Then, the system $\{T_{Ck}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{p\in\mathcal{P}} \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C^I m) = |\det C|\, \delta_{m,0} \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{4.6}$$

for every $m \in \mathbb{Z}^n$, where δ is the Kronecker delta in $\mathbb{R}^n$.

Proof. Since (4.6) follows immediately from (2.7) when $C_p = C$ for all $p \in \mathcal{P}$, then we only need to show that condition (2.6) is always satisfied in Theorem 2.1 under these conditions. Indeed, since $C_p = C$ for every $p \in \mathcal{P}$, then the sum with respect to m in (2.6) is finite (since $f \in \mathcal{D}$). If the system $\{T_{Ck}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a normalized tight frame, then, by Proposition 4.1,

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C|}\, |\hat{g}_p(\xi)|^2 \le 1. \tag{4.7}$$

Together with the fact that the sum with respect to m is finite, this implies (2.6). Similarly, if (4.6) holds, then we have inequality (4.7). Together with the fact that the sum with respect to m is finite, this implies (2.6), as in the previous case. □

5 Affine systems and wavelets

The classical **affine system** on $\mathbb{R}$ generated by $\psi \in L^2(\mathbb{R})$ is the collection

$$\mathcal{F}_2(\psi) = \{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : j \in \mathbb{Z}, k \in \mathbb{Z}\}. \tag{5.1}$$

This is the system (1.5) when the dimension is 1 and $A = 2$. As mentioned in Section 1, the characterization of those functions ψ for which $\mathcal{F}_2(\psi)$ is a normalized tight frame in $L^2(\mathbb{R})$ was accomplished by G. Gripenberg ([16]) and X. Wang ([35]), and this result has been extended to general dilations $a \in \mathbb{R}$, $a > 1$, (cf. [8, Th. 1]), and to $\mathbb{R}^n$ where dilations are performed by real expanding matrices (cf. [7, Cor. 2.4] and [24, Th. 5.1]).

To define these more general systems, we use the **translation operators** (as defined in Section 2) and the **dilation operators** D_A, $A \in GL_n(\mathbb{R})$, defined by

$$(D_A\, f)(x) = |\det A|^{1/2}\psi(Ax),$$

for $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Our affine systems will be generated by applying these operators to a finite family $\Psi = \{\psi^1, \dots \psi^L\} \subset L^2(\mathbb{R}^n)$, and they have the form

$$\mathcal{F}_A(\Psi) = \{D_A^j\, T_k\, \psi^\ell : j \in \mathbb{Z},\, k \in \mathbb{Z}^n,\, \ell = 1, \dots, L\}. \tag{5.2}$$

A simple calculation shows that

$$D_A^j\, T_k\, \psi^\ell = T_{A^{-j}k}\, D_A^j\, \psi^\ell\,,$$

so that, in order to apply Theorem 2.1, we are led to consider

$$\mathcal{P} = \big\{(j,\ell) : j \in \mathbb{Z},\ \ell = 1, 2, \ldots, L\big\},$$

$$g_p \equiv g_{(j,l)} = D_A^j\, \psi^\ell, \qquad \text{and} \qquad C_p \equiv C_{(j,\ell)} = A^{-j} \text{ for all } \ell = 1, \ldots, L.$$

There are good reasons for the fact that, in the literature, the characterizations of the systems $\mathcal{F}_A(\Psi)$, given by (5.2), that are normalized tight frames assume that the dilation matrices are **expanding**. In a private communication, D. Speegle has presented us with examples of dilation matrices which are not expanding for which there cannot exist any tight frame wavelets.

By definition, a matrix $M \in GL_n(R)$ is **expanding** on $\mathbb{R}^n$ if and only if all the eigenvalues of M have modulus greater than 1. There is an equivalent definition of expanding matrices (which we present in Lemma 5.2), that will be most useful for our purposes. To show this equivalence we need the following result:

Lemma 5.1. *Suppose* $M \in GL_n(\mathbb{R})$ *and* $\alpha, \beta \in \mathbb{R}$ *such that* $0 < \alpha < |\lambda| < \beta < \infty$ *for all eigenvalues* λ *of* M. *There exists* $C = C(M, \alpha, \beta) \geq 1$ *such that*

$$\frac{1}{C}\alpha^j |x| \leq |M^j\, x| \leq C\beta^j |x|, \tag{5.3}$$

when $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $j \geq 0$.

Remark. By applying (5.3) to $x = M^{-j}y$, we obtain

$$\frac{1}{C}\beta^{-j} |y| \leq |M^{-j}\, y| \leq C\alpha^{-j} |y|, \tag{5.4}$$

when $y \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $j \geq 0$.

Proof. We make use of the following fact involving the spectral radius, $\rho(M) = \max\{|\lambda| : \lambda \text{ eigenvalue of } M\}$:

$$\rho(M) = \lim_{n\to\infty} \|M^n\|^{1/n}$$

(see [31, p. 235]). Since $\rho(M) < \beta$, there exists $J_0 \in \mathbb{N}$ such that

$$|M^j\, x| \leq \|M^j\|\, |x| \leq \beta^j |x|$$

for $j \geq J_0$ and $x \in \mathbb{R}^n$. For $0 \leq j < J_0$ we have

$$|M^j\, x| \leq \|M^j\|\, |x| = \frac{\|M^j\|}{\beta^j}\beta^j\, |x| \leq \Big(\max_{0\leq j<J_0} \big\{\frac{\|M^j\|}{\beta^j}\big\}\Big)\beta^j\, |x|.$$

Hence, letting $C = \max_{0\leq j<J_0} \big\{1, \frac{\|M^j\|}{\beta^j}\big\}$ we have $|M^j\, x| \leq C\beta^j\, |x|$, for all $j \in \mathbb{Z}$, $j \geq 0$ and $x \in \mathbb{R}^n$. This gives us the right hand side inequality in (5.3). For the left hand side inequality of this formula,

apply the result just proved to $N = M^{-1}$; since $\rho(N) < 1/\alpha$, we deduce $|N^j\, y| \le C(1/\alpha)^j\, |y|$, for all $y \in \mathbb{R}^n$, $j > 0$, $j \in \mathbb{Z}$. The result follows by writing $y = M^j x$, for $x \in \mathbb{R}^n$. $\square$

Remark. Lemma (5.1) appears without proof in a paper by P.G. Lemarié-Rieusset ([26]). We thank G. Garrigós for pointing this reference to us.

Lemma 5.2. *A matrix $M \in GL_n(\mathbb{R})$ is expanding if and only if there exist $0 < k \le 1 < \gamma < \infty$ such that*

$$|M^j\, x| \ge k\gamma^j |x| \tag{5.5}$$

when $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $j \ge 0$. Moreover, if $M \in GL_n(\mathbb{R})$ is expanding, then we also have

$$|M^{-j}\, x| \le \frac{1}{k}\gamma^{-j}|x| \tag{5.6}$$

when $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $j \ge 0$.

Proof. If M is an expanding matrix, then we can choose $\alpha > 1$ in Lemma 5.1. Thus, (5.5) follows immediately from the left hand side inequality of (5.3). The inequality (5.6) follows by applying (5.5) to $y = M^{-j}x$.
Assume now that (5.5) holds and suppose that λ is an eigenvalue of M. If $\lambda \in \mathbb{R}$, let $x \in \mathbb{R}^n$ be an eigenvector corresponding to λ. By (5.5) we have

$$|\lambda|^j |x| = |\lambda^j x| = |M^j x| \ge k\gamma^j |x|$$

for all $j \in \mathbb{Z}$, $j \ge 0$. It follows that $|\lambda| \ge k^{1/j}\gamma$ for all $j \ge 0$. Hence, $|\lambda| \ge \gamma > 1$.

If $\lambda = \alpha + i\beta \in \mathbb{C}$, choose a corresponding eigenvector $u = x + iy \in \mathbb{C}^n$. Since M is expanding and $x \in \mathbb{R}^n$,

$$k\gamma^j\, |x| \le |M^j\, x| \le |M^j\, x + i\, M^j\, y| = |M^j(x + iy)| = |\lambda^j u| = |\lambda|^j |u|.$$

Without loss of generality we can assume $|y| \le |x|$, so that $|u| \le \sqrt{2}\,|x|$. It follows that $k\,\gamma^j\,|x| \le |\lambda|^j\,\sqrt{2}\,|x|$. Since $x \ne 0$ (otherwise $u = 0$), we have $k^{1/j}\,\gamma \le |\lambda|\,2^{1/j}$ for all $j \ge 0$. Hence, $1 < \gamma \le |\lambda|$. $\square$

The dilation matrices we are going to use are more general than the expanding ones: they could have some, but not all, of its eigenvalues with modulus 1, while the rest have modulus strictly larger than 1. Here we must notice that, if $|\det A| = 1$, then there is no $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ such that the Calderón condition:

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |\widehat{\psi^\ell}(B^{-j}\xi)|^2 = 1\,, \quad \text{for a. e. } \xi \in \mathbb{R}^n, \tag{5.7}$$

where $B = A^t$, holds. This result follows by an argument similar to one presented in [25], where continuous wavelets are studied. It is also known that, in some cases, when some of the eigenvalues of A have modulus greater than 1 and others have modulus smaller than 1, there is no

$\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ such that (5.7) hold, even if $|\det A| \neq 1$. As we pointed out before Lemma 5.1, this non-existence has been shown to us by D. Speegle, in a personal communication, for the case of diagonal dilation matrices. At the moment we are not aware of this (negative) result for general matrices A with these properties.

The dilation matrices we are going to use must have the properties described below:

Definition 5.1. Given $M \in GL_n(\mathbb{R})$ and a non-zero linear subspace F of $\mathbb{R}^n$, we say that M **is expanding on** F if there exists a complementary (not necessarily orthogonal) linear subspace E of $\mathbb{R}^n$ with the following properties:

(i) $\mathbb{R}^n = F + E$ and $F \cap E = \{0\}$;

(ii) $M(F) = F$ and $M(E) = E$, that is, F and E are invariant under M;

(iii) condition (5.5) (and therefore (5.6)) holds for all $x \in F$;

(iv) given $r \in \mathbb{N}$, there exists $C = C(M, r)$ such that, for all $j \in \mathbb{Z}$, the set

$$\mathcal{Z}_r^j(E) = \{m \in E \cap \mathbb{Z}^n : |M^j\, m| < r\}$$

has less than C elements.

Example 1. When M is an expanding matrix, Definition 5.1 is satisfied with $F = \mathbb{R}^n$ and $E = \{0\}$.

Example 2. For $a \in \mathbb{R}$, $|a| > 1$, the matrix

$$M = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

has eigenvalues a and 1. Letting F be the eigenspace corresponding to the eigenvalue a, and E the eigenspace corresponding to the eigenvalue 1, it is clear that M is expanding on F, in the sense of Definition 5.1. It is easy to obtain analogous, higher dimensional, diagonal matrices, even allowing some of the elements of the diagonal to be -1, that satisfy "expanding on F".

Example 3. More generally, given $a \in \mathbb{R}$, $|a| > 1$ and two independent vectors $u, v \in \mathbb{R}^2$, let M be a matrix for which u is an eigenvector corresponding to the eigenvalue a and v is an eigenvector corresponding to the eigenvalue 1. By taking $F = \{tu : t \in \mathbb{R}\}$ and $E = \{tv : t \in \mathbb{R}\}$ is easy to see that the conditions of Definition 5.1 are satisfied.

Example 4. For $a \in \mathbb{R}$, $|a| > 1$, and $\theta \in \mathbb{R}$, consider the matrix

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

which corresponds to a dilation on the X–axis and a rotation around the origin in the YZ–plane. The matrix M is expanding on $F = \mathbb{R} \times \{0\} \times \{0\}$, with $E = \{0\} \times \mathbb{R} \times \mathbb{R}$.

Example 5. For $a, b \in \mathbb{R}$, $|a| > 1$, consider

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

With $F = \mathbb{R} \times \{0\} \times \{0\}$, and $E = \{0\} \times \mathbb{R} \times \mathbb{R}$, properties (i), (ii), and (iii) of Definition 5.1 are obvious. A little bit of work is required to prove property (iv); however it is straightforward. Write $m \in E \cap \mathbb{Z}^3$ as $m = (0, m_2, m_3)$ with $m_2, m_3 \in \mathbb{Z}$. Since

$$M^j = \begin{pmatrix} a^j & 0 & 0 \\ 0 & 1 & jb \\ 0 & 0 & 1 \end{pmatrix}, \qquad j \in \mathbb{Z},$$

$|M^j\, m| < r$ implies $|m_2 + jbm_3|^2 + |m_3|^2 < r^2$. Hence $|m_3| < r$ and $|m_2 + jbm_3| < r$. For each $m_3 \in \mathbb{Z}$ fixed, there are at most $2r$ elements $m_2 \in \mathbb{Z}$ such that $|m_2 + jbm_3| < r$. Since there are at most $2r$ elements $m_3 \in \mathbb{Z}$ such that $|m_3| < r$, it follows that the number of elements in $\mathcal{Z}_r^j(E)$ does not exceed $4r^2$ for all $j \in \mathbb{Z}$.

The main result of this section is the following characterization of the affine systems $\mathcal{F}_A(\Psi)$, which will be obtained as a consequence of Theorem 2.1. As we mentioned in Section 1, this result is related and extends several other results that are in the literature. We will later show (Theorem 5.7) that there is an equivalent formulation of the following theorem, where (5.8) is replaced by a simpler expression.

Theorem 5.3. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. Then, the system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_\alpha} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + \alpha))} = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{5.8}$$

and all $\alpha \in \Lambda = \bigcup_{j\in\mathbb{Z}} B^j(\mathbb{Z}^n)$, where, for $\alpha \in \Lambda$, $\mathcal{P}_\alpha = \{j \in \mathbb{Z} : \, B^{-j}\alpha \in \mathbb{Z}^n\}$.

Proof. Recall that $D_A^j\, T_k\, \psi^\ell = T_{A^{-j}k}\, D_A^j\, \psi^\ell$. Apply Theorem 2.1 with

$$\mathcal{P} = \big\{(j, \ell) : j \in \mathbb{Z},\, \ell = 1, 2, \ldots, L\big\},$$

$$g_p \equiv g_{(j,l)} = D_A^j\, \psi^\ell, \qquad \text{and} \qquad C_p \equiv C_{(j,\ell)} = A^{-j} \text{ for all } \ell = 1, \ldots, L.$$

Since

$$\hat{g}_p(\xi) = (D_A^j \psi^\ell)^\wedge(\xi) = D_B^{-j}\, \hat{\psi}^\ell(\xi) = |\det B|^{-j/2} \hat{\psi}^\ell(B^{-j}\, \xi),$$

(5.8) follows from (2.7) in Theorem 2.1, provided the hypothesis (2.6) in this Theorem is satisfied. Therefore, all that it is left to prove is that the hypothesis (2.6) is satisfied in this particular case. Thus, we need to show that $L(f) < \infty$ for f in an appropriate dense set of $L^2(\mathbb{R}^n)$, where

$$L(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + B^j m)|^2\, |\det A^j|\, |(D_A^j \psi^\ell)^\wedge(\xi)|^2\, d\xi$$

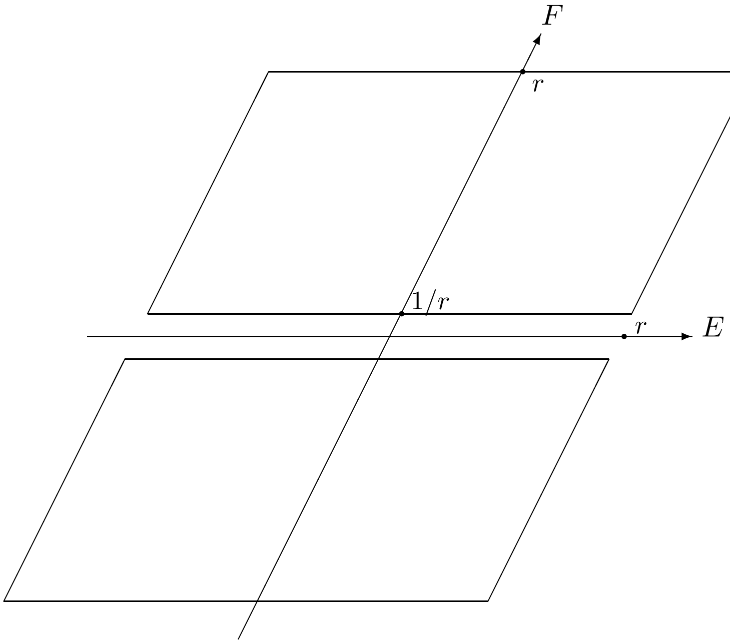

Figure 1: The set $Q(r)$ ($n = 2$).

$$= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\mathrm{supp}\,\hat{f}} |\hat{f}(\xi + B^j m)|^2 \, |\hat{\psi}^\ell(B^{-j}\xi)|^2 \, d\xi \,. \tag{5.9}$$

The dense set we choose is the following: since $B = A^t$ is expanding on F, we can then take E a complementary subspace to F as in Definition 5.1, and consider

$$D_E = \big\{ f \in L^2(\mathbb{R}^n) \colon \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \mathrm{supp}\,\hat{f} \text{ is compact in } \mathbb{R}^n \setminus E \big\} \tag{5.10}$$

where $\mathcal{D} = \{ f \in L^2(\mathbb{R}^n) \colon \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \mathrm{supp}\,\hat{f} \text{ is compact} \}$. This set $\mathcal{D}_E$ is dense in $L^2(\mathbb{R}^n)$, since E has measures zero.

The proof that $L(f) < \infty$ if $f \in \mathcal{D}_E$ is given in Proposition 5.6 below. To prove this delicate result we need some preparation and two lemmas.

Since B is expanding on F, by property (i) of Definition 5.1, given $x \in \mathbb{R}^n$, there exist unique $x_F \in F$ and $x_E \in E$ such that $x = x_F + x_E$. For $r, s \in \mathbb{R}$, $r > 1, s > 0$, define

$$Q(r,s) = \big\{ x = x_F + x_E : x_F \in F,\ x_E \in E,\ \frac{1}{r} < |x_F| < r,\ |x_E| < s \big\}, \tag{5.11}$$

and write $Q(r) = Q(r,r)$ (see Figure 5). It is clear that given any $f \in \mathcal{D}_E$ there exists $r \in \mathbb{N}$ such that $\mathrm{supp}\,\hat{f} \subset Q(r)$.

Lemma 5.4. *Let $M \in GL_n(\mathbb{R})$ be expanding on a subspace F of $\mathbb{R}^n$, and $r \in \mathbb{R}$. There exists $N = N(M,r) \in \mathbb{N}$ such that*

$$\#\{ j \in \mathbb{Z} : M^j \eta \in Q(r) \} \le N$$

for all $\eta \in \mathbb{R}^n$.

Proof. Choose E to be a complementary subspace for F as in Definition 5.1. If $\eta \in E$, we can choose $N = 1$. For $\eta \notin E$, write $\eta = \eta_F + \eta_E$ with $\eta_F \in F$, $\eta_E \in E$ and $\eta_F \neq 0$. Then, by (ii) of Definition 5.1, for any $j \in \mathbb{Z}$, we have $M^j\eta = M^j\eta_F + M^j\eta_E$ with $M^j\eta_F \in F$ and $M^j\eta_E \in E$. Choose $j_0 = j_0(\eta)$ to be the smallest integer such that $|M^{j_0}\eta_F| > 1/r$. This is possible since, by property (iii) of the matrix M, there exist $0 < k \leq 1 < \gamma < \infty$ such that

$$|M^j\eta_F| \geq k\gamma^j|\eta_F| \quad \text{if} \quad j \in \mathbb{Z},\, j \geq 0\,,$$

and

$$|M^{-j}\eta_F| \leq \frac{1}{k}\gamma^{-j}|\eta_F| \quad \text{if} \quad j \in \mathbb{Z},\, j \geq 0\,.$$

Thus, if $j < j_0$, then $|M^j\eta_F| \leq 1/r$, which implies that $M^j\eta \notin Q(r)$ by the definition of $Q(r)$. Choose $N_0 = 1+[\log_\gamma(r^2/k)]$ (observe that $k/r \leq 1/r < r$ implies $r^2/k > 1$, so that $[\log_\gamma(r^2/k)] \geq 0$). Since M is expanding on F, if $j \geq N_0 \geq 1$, we have

$$|M^{j+j_0}\eta_F| = |M^j M^{j_0}\eta_F| \geq k\gamma^j|M^{j_0}\eta_F| > k\gamma^j\frac{1}{r}\,,$$

by the choice of j_0. Thus,

$$|M^{j+j_0}\eta_F| > k\gamma^j\frac{1}{r} \geq k\gamma^{N_0}\frac{1}{r} \geq k\frac{r^2}{k}\frac{1}{r} = r\,.$$

This shows that if $j \geq N_0$, then $M^{j+j_0}\eta_F \notin Q(r)$. Hence,

$$\{j \in \mathbb{Z} : M^j\eta \in Q(r)\} \subset \{j_0, j_0+1, \cdots, j_0+N_0-1\}.$$

By taking $N = N_0$ the proof is finished. $\square$

Remark. Lemma 5.4 is adapted from [2, Lemma 2.3], where the result is proved only for expanding matrices on $\mathbb{R}^n$.

For $r, s \in \mathbb{R}$, define

$$\widetilde{Q}(r,s) = \{x = x_F + x_E : x_F \in F,\, x_E \in E,\, |x_F| < r,\, |x_E| < s\}\,,$$

and write $\widetilde{Q}(r) = \widetilde{Q}(r,r)$. These sets will be used in the statement and the proof of the next lemma.

Lemma 5.5. *Let* $M \in GL_n(\mathbb{R})$ *be expanding on a subspace* F *of* $\mathbb{R}^n$, $r \in \mathbb{R}$, *and* E *be a complementary subspace of* F *as in Definition 5.1. There exists* $\widetilde{C} = \widetilde{C}(M,r) \in \mathbb{R}$ *such that*

$$\#\{m \in \mathbb{Z}^n \setminus E : M^j m \in \widetilde{Q}(r)\} \leq \widetilde{C}|\det M|^{-j}$$

for all $j \in \mathbb{Z}$.

Proof. For $m \in \mathbb{Z}^n \setminus E$, write $m = m_F + m_E$ with $m_F \in F$, $m_E \in E$ and $m_F \neq 0$. Let

$$T_r = \inf\{|m_F| : m \in (\mathbb{Z}^n \setminus E) \cap \widetilde{Q}(r)\} > 0.$$

Take j_1 to be the smallest positive integer greater than $\log_\gamma(r/(kT_r))$, where k and γ are as in Lemma 5.2 (adapted to Definition 5.1). If $j \geq j_1$ and $m \in \mathbb{Z}^n \setminus E$, then, by (iii) of Definition 5.1, we have $|M^j m_F| \geq k\gamma^j |m_F| \geq k\frac{r}{kT_r}T_r = r$. Hence, for $j \geq j_1$,

$$\#\{m \in \mathbb{Z}^n \setminus E : M^j m \in \widetilde{Q}(r)\} = 0. \tag{5.12}$$

Thus, we only need to consider $j < j_1$. Choose $m \in \mathbb{Z}^n \setminus E$ with $M^j m \in \widetilde{Q}(r)$, $\xi \in [0,1)^n$ and $j < j_1$. Write $\xi = \xi_F + \xi_E$ with $\xi_F \in F$ and $\xi_E \in E$. Since M is expanding on F,

$$\begin{aligned}|M^{-j_1+j}(m_F + \xi_F)| \leq& |M^{-j_1+j}(m_F)| + |M^{-j_1+j}(\xi_F)| \\ \leq& \frac{1}{k}\gamma^{-j_1}|M^j(m_F)| + \frac{1}{k}\gamma^{-j_1+j}|\xi_F| \\ <& \frac{1}{k}\gamma^{-j_1}r + \frac{1}{k}|\xi_F| \leq \frac{1}{k}\gamma^{-j_1}r + \frac{1}{k}S_1 \equiv R_1\end{aligned}$$

where $S_1 = \sup\{|\xi_F| : \xi \in [0,1)^n\}$. Also, since $\|M\| \geq \rho(M) \geq 1$, we have

$$|M^{-j_1+j}(m_E + \xi_E)| \leq \|M\|^{-j_1}r + \|M\|^{-j_1+j}|\xi_E| \leq \|M\|^{-j_1}r + S_2 \equiv R_2,$$

where $S_2 = \sup\{|\xi_E| : \xi \in [0,1)^n\}$. We have just shown that

$$\{m \in \mathbb{Z}^n \setminus E : M^j\eta \in \widetilde{Q}(r)\} \subset \{m \in \mathbb{Z}^n : m + [0,1)^n \subset M^{j_1-j}(\widetilde{Q}(R_1,R_2))\} \equiv \mathcal{M}^j_{R_1,R_2}.$$

Since the sets $m + [0,1)^n$, $m \in \mathbb{Z}^n$, are disjoint,

$$\begin{aligned}\#\{m \in \mathbb{Z}^n \setminus E : M^j\eta \in \tilde{Q}(r)\} \leq \#\mathcal{M}^j_{R_1,R_2} = \Big|\bigcup_{m\in\mathcal{M}^j_{R_1,R_2}} (m + [0,1)^n)\Big| \\ \leq \big|M^{j_1-j}(\widetilde{Q}(R_1,R_2))\big| = |\widetilde{Q}(R_1,R_2)|\,|\det M|^{j_1}\,|\det M|^{-j}\end{aligned} \tag{5.13}$$

The Lemma then follows from (5.12) and (5.13) by taking $\widetilde{C} = |\widetilde{Q}(R_1,R_2)|\,|\det M|^{j_1}$. $\square$

We can now go back to our task of showing that $L(f) < \infty$. The situation here is different from the case we encountered in Section 3, where we showed that the integrability condition (2.6) follows from the fact that $g \in L^2(\mathbb{R}^n)$ (recall that, in the case of Gabor systems, the matrices C_p are independent of p). We will show in the following Proposition that if

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 \leq 1 \qquad \text{for a. e.} \qquad \xi \in \mathbb{R}^n, \tag{5.14}$$

then the integrability condition (2.6) is satisfied for the affine system $\mathcal{F}_A$ (where the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$). Observe that if $\mathcal{F}_A(\Psi)$ is a normalized tight frame for $L^2(\mathbb{R}^n)$, then, by Proposition 4.1 applied to the affine system (5.2), we deduce inequality (5.14). This inequality also holds if we assume (5.8) (take $\alpha = 0$). Therefore, the following Proposition implies that if either $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame or if the case $\alpha = 0$ of (5.8) holds, then $L(f) < \infty$, where $L(f)$ is given by (5.9). This is all we need to finish the proof of Theorem 5.3.

Proposition 5.6. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. If (5.14) holds,*

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 \le 1 \qquad \textit{for a. e.} \qquad \xi \in \mathbb{R}^n, \tag{5.15}$$

then $L(f) < \infty$, where $L(f)$ is given by (5.9).

Proof. Let $f \in \mathcal{D}_E$ and choose $r \in \mathbb{N}$ such that $\operatorname{supp} \hat{f} \subset Q(r)$. Then

$$\begin{aligned} L(f) &\le \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{Q(r)} |\hat{f}(\xi + B^j m)|^2 \, |\hat{\psi}^\ell(B^{-j}\xi)|^2 \, d\xi \\ &= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{B^j\eta\in Q(r)} |\hat{f}(B^j(\eta + m)|^2 \, |\hat{\psi}^\ell(\eta)|^2 \, |\det B|^j \, d\eta \,. \end{aligned} \tag{5.16}$$

We write $L_0(f)$ for the sum of the terms in (5.16) for which $m \in E \cap \mathbb{Z}^n$, and $L_1(f)$ for the sum of the terms in the same expression for which $m \in \mathbb{Z}^n \setminus E$. Then, $L(f) = L_0(f) + L_1(f)$.

We first estimate $L_0(f)$. For $m \in E \cap \mathbb{Z}^n$, if $\xi \in Q(r)$ and $\xi + B^j m \in Q(r)$, then, for $j \in \mathbb{Z}$, we have

$$|B^j m| \le |\xi_E + B^j m| + |\xi_E| < r + r = 2r\,,$$

where $\xi = \xi_F + \xi_E$, with $\xi_F \in F$ and $\xi_E \in E$. Thus, using the notation introduced in property (iv) of Definition 5.1, we have:

$$\{m \in E \cap \mathbb{Z}^n : \xi \in Q(r) \text{ and } \xi + B^j m \in Q(r)\} \subset \mathcal{Z}^j_{2r}(E),$$

for every $j \in \mathbb{Z}$. By property (iv) of Definition 5.1, the number of elements in $\mathcal{Z}^j_{2r}(E)$ is less than $C = C(B, 2r)$ for all $j \in \mathbb{Z}$. Thus

$$L_0(f) \le C(B, 2r)\|\hat{f}\|^2_\infty \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \int_{Q(r)} |\hat{\psi}^\ell(B^{-j}\xi)|^2 \, d\xi.$$

Using (5.14), it follows that

$$L_0(f) \le C(B, 2r)\|\hat{f}\|^2_\infty |Q(r)| < \infty. \tag{5.17}$$

We now estimate $L_1(f)$. For $m \in \mathbb{Z}^n \setminus E$, if $B^j\eta \in Q(r)$ and $B^j(\eta + m) \in Q(r)$, then, for $j \in \mathbb{Z}$, we have that

$$|B^j m_F| \le |B^j(\eta_F + m_F)| + |B^j \eta_F| < r + r = 2r,$$

and

$$|B^j m_E| \le |B^j(\eta_E + m_E)| + |B^j \eta_E| < r + r = 2r,$$

where we decomposed m and η as a unique sum of elements in F and E. Thus, with the notation introduced before Lemma 5.5,

$$\{m \in \mathbb{Z}^n \setminus E : B^j\eta \in Q(r) \text{ and } B^j(\eta + m) \in Q(r)\} \subset \{m \in \mathbb{Z}^n \setminus E : B^j m \in \widetilde{Q}(2r)\},$$

for every $j \in \mathbb{Z}$. By Lemma 5.5, the number of elements in the last set does not exceed $\widetilde{C}(B,2r)|\det B|^{-j}$, for all $j \in \mathbb{Z}$. Thus,

$$L_1(f) \le \widetilde{C}(B,2r)\|\hat{f}\|_\infty^2 \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\int_{B^j\eta\in Q(r)} |\hat{\psi}^\ell(\eta)|^2\,d\eta.$$

By Lemma 5.4, the number of $j \in \mathbb{Z}$ such that $B^j\eta \in Q(r)$ does not exceed a fixed number, $N(B,r)$, independently of $\eta \in \mathbb{R}^n$. Hence,

$$L_1(f) \le \widetilde{C}(B,2r)\|\hat{f}\|_\infty^2\, N(B,r) \sum_{\ell=1}^{L} \|\hat{\psi}^\ell\|_2^2 < \infty. \tag{5.18}$$

From (5.16), (5.17), and (5.18) we deduce that, if $f \in \mathcal{D}_E$, then $L(f) < \infty$. □

Equality (5.8) in Theorem 5.3 can be written in a simpler form involving the lattice points $m \in \mathbb{Z}^n$ instead of the elements $\alpha \in \Lambda$. This shows that there is a redundancy in the original condition (5.8). We have the following:

Theorem 5.7. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. Then the system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{\ell=1}^{L}\sum_{j\in\mathcal{P}_m} \hat{\psi}^\ell(B^{-j}\xi)\,\overline{\hat{\psi}^\ell(B^{-j}(\xi+m))} = \delta_{m,0} \quad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{5.19}$$

and all $m \in \mathbb{Z}^n$, where $\mathcal{P}_m = \{j \in \mathbb{Z} :\ B^{-j}m \in \mathbb{Z}^n\}$.

Proof. It is enough to show that (5.8) is valid for each $\alpha \in \Lambda$ if and only if (5.19) is valid for each $m \in \mathbb{Z}^n$. Each lattice point $m \in \mathbb{Z}^n$ belongs to Λ since $\mathbb{Z}^n = B^0(\mathbb{Z}^n) \subset \Lambda = \cup_{j\in\mathbb{Z}} B^j(\mathbb{Z}^n)$, and, therefore, (5.8) implies (5.19). Now, suppose that (5.19) is true for all $m \in \mathbb{Z}^n \setminus \{0\}$ (the case $m = 0$ in (5.19) is equal to the case $\alpha = 0$ in (5.8), and so we only have to consider the case $m \neq 0$). For any $\alpha \in \Lambda \setminus \{0\}$, we have $\alpha = B^{j_0}m_0$ for some $j_0 \in \mathbb{Z}$ and some $m_0 \in \mathbb{Z}^n \setminus \{0\}$. By making the change of variables $\xi = B^{j_0}\eta$ in the left hand side of (5.8), we obtain

$$\sum_{\ell=1}^{L}\sum_{j\in\mathcal{P}_\alpha} \hat{\psi}^\ell(B^{-j}\xi)\,\overline{\hat{\psi}^\ell(B^{-j}(\xi+\alpha))} = \sum_{\ell=1}^{L}\sum_{j\in\mathcal{P}_{B^{j_0}m_0}} \hat{\psi}^\ell(B^{-j+j_0}\eta)\,\overline{\hat{\psi}^\ell(B^{-j+j_0}(\eta+m_0))} \tag{5.20}$$

Let $k = j - j_0$ and observe that $\mathcal{P}_\alpha = \mathcal{P}_{B^{j_0}m_0} = \{j \in \mathbb{Z} : B^{-j}(B^{j_0}m) \in \mathbb{Z}^n\}$. Since $B^{-(k+j_0)}(B^{j_0}m_0) = B^{-k}m_0$, it follows that $j = k + j_0 \in \mathcal{P}_{B^{j_0}m_0}$ if and only if $k \in \mathcal{P}_{m_0}$. Replacing $-j + j_0$ by $-k$ in the second sum of (5.20), we obtain

$$\sum_{\ell=1}^{L}\sum_{j\in\mathcal{P}_\alpha} \hat{\psi}^\ell(B^{-j}\xi)\,\overline{\hat{\psi}^\ell(B^{-j}(\xi+\alpha))} = \sum_{\ell=1}^{L}\sum_{k\in\mathcal{P}_{m_0}} \hat{\psi}^\ell(B^{-k}\eta)\,\overline{\hat{\psi}^\ell(B^{-k}(\eta+m_0))}$$

and the last expression is zero for a.e. $\eta \in \mathbb{R}^n$ by (5.19) (recall that $m_0 \neq 0$). So the left hand side is also zero for a.e. $\xi \in \mathbb{R}^n$ when $\alpha \in \Lambda \setminus \{0\}$. □

Examples of **orthonormal A-wavelets** (that is, systems $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$, such that $\mathcal{F}_A(\Psi)$ is an orthonormal basis of $L^2(\mathbb{R}^n)$) for expanding matrices on $\mathbb{R}^n$ can be found in [13, 34, 14]. Here we show how Theorem 5.3 can be applied to obtain examples of **orthonormal A-wavelets** for some dilation matrices A for which $B = A^t$ satisfy Definition 5.1, but is not necessarily expanding on $\mathbb{R}^n$.

In order to construct these examples, observe that if $j \in \mathcal{P}_k$ (see Theorem 5.7 for the definition of the set $\mathcal{P}_k$ we use here), then $B^{-j}k = m \in \mathbb{Z}^n$, so that if $\psi^\ell \in L^2(\mathbb{R}^n)$ and $(\text{supp}\,\hat{\psi}^\ell) \cap (\text{supp}\,\hat{\psi}^\ell(\cdot - m)) = \emptyset$ (a.e) for all $m \in \mathbb{Z}^n \setminus \{0\}$, $\ell = 1, \cdots, L$, then all the equations in (5.19) with $k \neq 0$ are trivially true. Since $\mathcal{P}_0 = \mathbb{Z}$, we have the following:

Corollary 5.8. *Assume the same set up as in Theorem 5.7, and suppose that* $(\text{supp}\,\hat{\psi}^\ell) \cap (\text{supp}\,\hat{\psi}^\ell(\cdot - m)) = \emptyset$ *(a.e) for all* $m \in \mathbb{Z}^n \setminus \{0\}$, $\ell = 1, \cdots, L$. *If*

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 = 1 \quad \text{for a. e} \quad \xi \in \mathbb{R}^n, \tag{5.21}$$

then the system $\mathcal{F}_A(\Psi)$ *is a normalized tight frame for* $L^2(\mathbb{R}^n)$. *If, in addition,* $\|\psi^\ell\|_2 = 1$ *for all* $\ell = 1, \cdots, L$, *then* $\Psi = \{\psi^1, \cdots, \psi^L\}$ *is an orthonormal A-wavelet for* $L^2(\mathbb{R}^n)$.

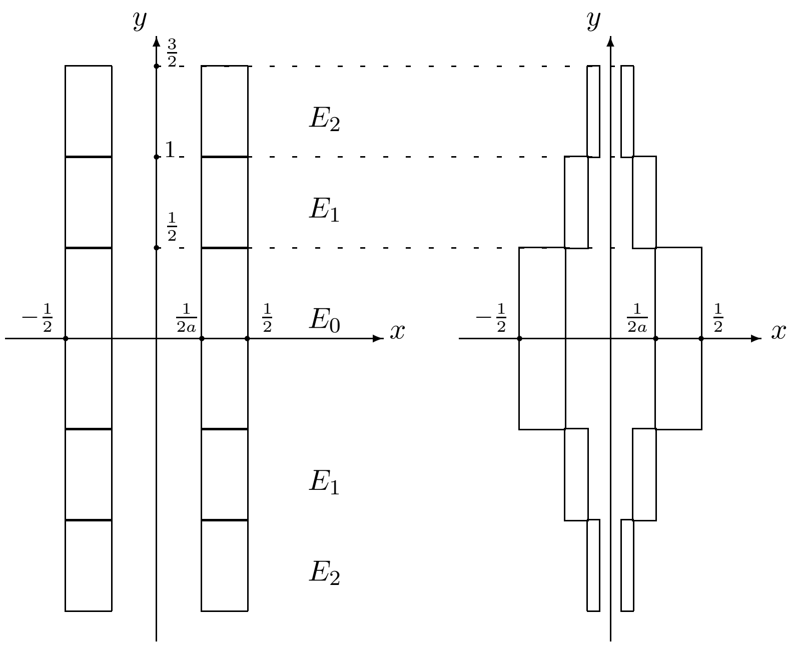

Figure 2: Example 6.

Example 6. For $a \in \mathbb{R}, a > 1$, let

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

as in Example 2. We construct a single function $\psi \in L^2(\mathbb{R}^2)$, with $\|\psi\|_2 = 1$, such that ψ is an orthonormal A-wavelet. The vertical strips

$$V = \{(x,y) \in \mathbb{R}^2 : \frac{1}{2a} \le |x| < \frac{1}{2}\}$$

are tiled by the sets

$$E_n = \{(x,y) \in \mathbb{R}^2 : \frac{1}{2a} \le |x| < \frac{1}{2}, \frac{n}{2} \le |y| < \frac{n+1}{2}\} \quad n = 0, 1, 2, \dots$$

(see Figure 2). Define

$$S_n = A^{-n} E_n, \quad n = 0, 1, 2, \dots \qquad \text{and} \qquad W = \bigcup_{n=0}^{\infty} S_n .$$

Observe that W is a disjoint union of the sets S_n . Thus, we have

$$|W| = \sum_{n=0}^{\infty} 4 \frac{1}{2} (\frac{1}{2a^n} - \frac{1}{2a^{n+1}}) = (1 - \frac{1}{a}) \sum_{n=0}^{\infty} \frac{1}{a^n} = 1 \, .$$

Define $\psi \in L^2(\mathbb{R}^2)$ by $\hat{\psi} = \chi_W$. The above computation shows that $\|\psi\|_2 = 1$. Since $\bigcup_{n=0}^{\infty} A^n S_n = \bigcup_{n=0}^{\infty} E_n = V$, and $\{A^j V : j \in \mathbb{Z}\}$ is a tiling of $\mathbb{R}^2$ by the vertical strips $A^j V$, (5.21) follows. Finally, observe that horizontal and vertical translations of W by non zero integers do not overlap. Hence, ψ is an orthonormal A-wavelet. (An example similar to this one has been exhibited in [5] for the case $a = 2$.)

Remarks

(**1**) Applying Theorem 4.2 to the affine system $\mathcal{F}_A(\Psi)$, it follows that $\mathcal{F}_A(\Psi)$ is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if it is a Bessel system with constant 1 and the Calderón condition (5.21) holds. In particular, if $\mathcal{F}_A(\Psi)$ is an orthonormal system, then $\mathcal{F}_A(\Psi)$ is complete if and only if (5.21) holds. See Remark (1) following Theorem 4.2 for appropriate references to this result.

(**2**) The ideas presented in this section apply to more general affine systems than (5.2). For $\{\psi^1, \cdots, \psi^L\} \in L^2(\mathbb{R}^n)$, $A_1, \dots, A_L \in GL_n(\mathbb{R})$ and $N_1, \dots N_L \in GL_n(\mathbb{R})$, consider the affine systems

$$\{D^j_{A_\ell} \, T_{N_\ell k} \, \psi^\ell : j \in \mathbb{Z}, \, k \in \mathbb{Z}^n, \, \ell = 1, \dots, L\} \, . \tag{5.22}$$

Since $D^j_{A_\ell} \, T_{N_\ell k} \, \psi^\ell(x) = T_{A_\ell{}^{-j} N_\ell k} \, D^j_{A_\ell} \, \psi^\ell(x)$, the system (5.22) can be described as a collection of the form (2.1) for appropriate choices of $\mathcal{P}$, g_p and C_p. Then Theorems 2.1 and 4.2 can be applied to characterize normalized tight frames for the affine systems given by (5.22). Since the study of this case is very similar to Theorem 5.3, the details will be omitted. The results one obtains in the case of expansive dilation matrices $A_1, \dots, A_L \in GL_n(\mathbb{R})$ can be found in [24].

6 Affine systems and wavelets: special dilation matrices

In this section, we are going to analyze the forms that the characterization equations (5.19) assume for different values of $m \in \mathbb{Z}^n$, depending on the dilation matrices A: for example, a corollary of our work in this section is that, for affine systems in one dimension with $A = 2$, the equations (5.19) in Theorem 5.7 are the equations (1.6) and (1.7) in the classical Theorem 1.1.

Observe that the major difference between these two equations is that, in the first, we encounter the sum over all $j \in \mathbb{Z}$, while, in the second, the sum is over all $j \geq 0$. In terms of the notation used for the general case in (5.19), (1.6) and (1.7) represent the two types of equations obtained when $m = 0$ (the Calderón condition we already discussed) and the case when $m \neq 0$.

We will present different classes of dilation matrices where there are, in fact, three or more types of equalities. We always have the case $m = 0$, which, in terms of the notation in Theorem 5.7), gives $\mathcal{P}_0 = \mathbb{Z}$ and represents the Calderón condition

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 = 1 \quad \text{for a. e.} \quad \xi \in \mathbb{R}^n \,. \tag{6.1}$$

As we shall see, the case $m \neq 0$ can assume several different forms. In the simple example $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, there are two different types of equalities besides the case corresponding to $m = 0$. If $m = \begin{pmatrix} 0 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2$, $m_2 \neq 0$, we have $\mathcal{P}_m = \mathbb{Z}$ since $A^{-j}k = \begin{pmatrix} 0 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2$ for all $j \in \mathbb{Z}$. On the other hand, if m is not of the above form, then one obtains an equation similar to equation (1.7) (see Example 8 for details).

To better understand how these cases arise, let us first consider the intersection

$$I(B) = \bigcap_{i\in\mathbb{Z}} B^i(\mathbb{Z}^n),$$

where $B = A^t \in GL_n(\mathbb{R})$, and B is expanding on a subspace F of $\mathbb{R}^n$.
If B is expanding on $\mathbb{R}^n$, then $I(B) = \{0\}$. In general, $I(B) \subset \mathbb{Z}^n$. When $I(B)$ is not empty and $m \in I(B)$, $m \neq 0$, then we have $\mathcal{P}_m = \mathbb{Z}$, and (5.19) is equivalent to

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + m))} = 0 \quad \text{for a. e.} \quad \xi \in \mathbb{R}^n \,. \tag{6.2}$$

Observe that (6.2) is void for dilation matrices expanding on $\mathbb{R}^n$. For the matrices of Example 2, we have $I(B) = \{0\} \times \mathbb{Z}$. For the matrices of Example 3, we have $I(B) = E \cap \mathbb{Z}^2$. For the matrices of Example 4, the set $I(B)$ depends on the angle of rotation θ. For the matrices of Example 5, we have $I(B) = \{0\} \times \mathbb{Z} \times \mathbb{Z}$ when b is an integer.

We describe further how equation(5.19) assumes different forms by selecting three types of the dilation matrix $B = A^t \in GL_n(\mathbb{R})$.

6.1 Matrices of Type-I

Definition 6.1. A matrix $M \in GL_n(\mathbb{R})$ is of **Type-I** if

$$M^j(\mathbb{Z}^n) \cap \mathbb{Z}^n = I(M) = \bigcap_{i \in \mathbb{Z}} M^i(\mathbb{Z}^n) \qquad (6.3)$$

for all $j \in \mathbb{Z} \setminus \{0\}$.

Examples of matrices of Type-I are the matrices $M = aI_n$ with $a \in \mathbb{R}$ such that $a^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z} \setminus \{0\}$; in this case, $I(M) = \{0\}$. More generally, any diagonal matrix whose diagonal entries a_{ii} are such that $a_{ii}^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z} \setminus \{0\}$ is a matrix of Type-I. The matrices of Example 2 are also of this type when $a \in \mathbb{R}$ is such that $a^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z} \setminus \{0\}$; in this case $I(M) = \{0\} \times \mathbb{Z}$.

We now apply Theorem 5.7 to characterize the affine system $\mathcal{F}_A(\Psi)$, given by (5.2), where $B = A^t$ is a matrix of Type-I that is expanding on a subspace of $\mathbb{R}^n$.

Proposition 6.1. *Let $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that $B = A^t$ is a matrix of Type-I which is expanding on a subspace of $\mathbb{R}^n$. Then the affine system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions hold: (6.1), (6.2) and*

$$\sum_{\ell=1}^{L} \hat{\psi}^\ell(\xi)\, \overline{\hat{\psi}^\ell(\xi + m)} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \qquad (6.4)$$

and all $m \in \mathbb{Z}^n \setminus I(B)$.

Proof. We have already observed that (6.1) and (6.2) correspond to the cases $m = 0$ and $m \in I(B) \setminus \{0\}$ of (5.19). Thus, we only need to consider the case $m \in \mathbb{Z}^n \setminus I(B)$. It is clear that the set $\mathcal{P}_m = \{j \in \mathbb{Z} : B^{-j}m \in \mathbb{Z}^n\}$ contains the element $j = 0$. We now show that it does not contain any other element. If there exist $j \in \mathbb{Z}$, with $j \neq 0$, such that $j \in \mathcal{P}_m$, then we must have $B^{-j}m \in \mathbb{Z}^n$. Since $-j \neq 0$ we deduce from (6.3) that $m \in I(B)$, contrary to the properties of m. Hence, $\mathcal{P}_m = \{0\}$ and (5.19) gives

$$\sum_{\ell=1}^{L} \hat{\psi}^\ell(\xi)\overline{\hat{\psi}^\ell(\xi + m)} = 0 \qquad \text{for a. e.} \qquad \xi \in \mathbb{R}^n,$$

which is what we wanted to prove. $\square$

Example 7. For the matrix $A = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$ and a single $\psi \in L^2(\mathbb{R}^2)$, it follows from Proposition 6.1 that the affine system $\mathcal{F}_A(\psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^2)$ if and only if

$$\sum_{j \in \mathbb{Z}} |\, \hat{\psi}(\pi^j \xi_1,\ \xi_2)\, |^2 = 1 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R},$$

$$\sum_{j \in \mathbb{Z}} \hat{\psi}(\pi^j \xi_1,\ \xi_2)\, \overline{\hat{\psi}(\pi^j \xi_1,\ \xi_2 + m_2)} = 0 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R}, \text{ and all } m_2 \in \mathbb{Z} \setminus \{0\},$$

$$\hat{\psi}(\xi_1,\ \xi_2)\, \overline{\hat{\psi}(\xi_1 + m_1,\ \xi_2 + m_2)} = 0 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R}, \text{ and all } m_1 \in \mathbb{Z} \setminus \{0\}, m_2 \in \mathbb{Z}.$$

6.2 Matrices of Type-II

Definition 6.2. A matrix $M \in GL_n(\mathbb{R})$ is of **Type-II** if $M(\mathbb{Z}^n) \subset \mathbb{Z}^n$ (equivalently, all the entries of M are integers).

Before we describe the equations that characterize affine systems for dilation matrices of Type-II that are expanding on a subspace of $\mathbb{R}^n$, we make the following observation.

Lemma 6.2. *Let $M \in GL_n(\mathbb{Z})$. If $m \in \mathbb{Z}^n \setminus I(M)$, there exist unique $d \in \mathbb{Z}^+ \cup \{0\}$ and $r \in \mathbb{Z}^n \setminus M(\mathbb{Z}^n)$ such that $m = M^d r$.*

Proof. If $m \notin M(\mathbb{Z}^n)$, then write $m = M^0 m$ and the result follows by taking $d = 0$, $r = m$. If $m \in M(\mathbb{Z}^n)$, write $m = Mm_1$ with $m_1 \in \mathbb{Z}^n$; while, if $m_1 \notin M(\mathbb{Z}^n)$, the result follows by taking $d = 1$, $r = m_1$. If $m_1 \in M(\mathbb{Z}^n)$, write $m_1 = Mm_2$ with $m_2 \in \mathbb{Z}^n$. Thus, $m = M^2 m_2$. If $m_2 \notin M(\mathbb{Z}^n)$, the result follows by taking $d = 2$, $r = m_2$. Continue in this way. This process stops. Otherwise, $m = M^j m_j$ for all $j \in \mathbb{Z}^+$, with $m_j \in \mathbb{Z}^n$. Since,

$$\cdots \subset M^2(\mathbb{Z}^n) \subset M(\mathbb{Z}^n) \subset \mathbb{Z}^n \subset M^{-1}(\mathbb{Z}^n) \subset M^{-2}(\mathbb{Z}^n) \subset \cdots, \tag{6.5}$$

we deduce $m \in I(M)$, contrary to our assumption.
To show uniqueness, suppose that $M^d r = m = M^{d_1} m_1$ with $d_1 \geq d$ and $r, r_1 \in \mathbb{Z}^n \setminus M(\mathbb{Z}^n)$. Then, $r = M^{d_1 - d} m_1$. Since $r \notin M(\mathbb{Z}^n)$, we deduce from (6.5) that $r \notin M^{d_1-d}(\mathbb{Z}^n)$ if $d_1 > d$. Hence, $d_1 = d$ and $r = r_1$. □

Proposition 6.3. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and let $A \in GL_n(\mathbb{R})$ be such that $B = A^t$ is a matrix of Type-II which is expanding on a subspace of $\mathbb{R}^n$. Then the affine system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions hold: (6.1), (6.2) and*

$$\sum_{\ell=1}^{L} \sum_{j \geq 0} \hat{\psi}^\ell(B^j \xi)\, \overline{\hat{\psi}^\ell(B^j(\xi + r))} = 0 \quad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{6.6}$$

and all $r \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$ (observe that $r \notin I(B)$).

Proof. We have already observed that (6.1) and (6.2) are the cases $m = 0$ and $m \in I(B) \setminus \{0\}$ of (5.19). Thus, we only need to consider the case $m \in \mathbb{Z}^n \setminus I(B)$.

We want to examine $\mathcal{P}_m = \{j \in \mathbb{Z} : B^{-j} m \in \mathbb{Z}^n\}$. If $j \in \mathcal{P}_m$, we have $B^{-j} m = s \in \mathbb{Z}^n$. By Lemma 6.2, there exist unique $d \in \mathbb{Z}^+ \cup \{0\}$ and $r \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$ such that $m = B^d r$. Hence, $s = B^{-j+d} r$. We must have $-j + d \geq 0$ (otherwise, with $-j + d = -\ell < 0$, we deduce $s = B^{-\ell} r$, and $r = B^\ell s \in B(\mathbb{Z}^n)$).

Thus, for $m = B^d r \in \mathbb{Z}^n \setminus I(B)$, (5.19) of Theorem 5.7 is equivalent to

$$\sum_{\ell=1}^{L} \sum_{j \leq d} \hat{\psi}^\ell(B^{-j} \xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + B^d r))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{6.7}$$

with $r \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$. Replacing ξ by $B^d \eta$ in the above expression and then changing the index of summation to $k = d - j$ we obtain (6.6). □

Example 8. For the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and a single $\psi \in L^2(\mathbb{R}^2)$, it follows from Proposition 6.3 that the affine system $\mathcal{F}_A(\psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^2)$ if and only if

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi_1,\ \xi_2)|^2 = 1 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R},$$

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(2^j\xi_1,\ \xi_2)\,\overline{\hat{\psi}(2^j\xi_1,\ \xi_2+m_2)} = 0 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R}, \text{and all } m_2 \in \mathbb{Z}\setminus\{0\},$$

$$\sum_{j\geq 0} \hat{\psi}(2^j\xi_1,\ \xi_2)\,\overline{\hat{\psi}(2^j(\xi_1+q_1),\ \xi_2+m_2)} = 0 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R}, \text{and all } q_1 \in \mathbb{Z}\setminus 2\mathbb{Z}, m_2 \in \mathbb{Z}.$$

6.3 Matrices of Type-III

Definition 6.3. A matrix $M \in GL_n(\mathbb{R})$ is of **Type-III** if there exists $\delta \in \mathbb{N}$, $\delta > 1$, such that:

$$\text{(i) } M^\delta(\mathbb{Z}^n) \subset \mathbb{Z}^n,$$

and

$$\text{(ii) } M^r(\mathbb{Z}^n) \cap \mathbb{Z}^n = I(M) = \bigcap_{i\in\mathbb{Z}} B^i(M) \quad \text{for all} \quad 0 < r < \delta,\ r \in \mathbb{Z}.$$

The following are examples of matrices of Type-III:

$$M_1 = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Observe that the lower-right 2×2 matrix of M_3 is a rotation by $\pi/4$ radians. Since M_1^2, M_2^2, and M_3^2 are matrices with integer entries, the matrices M_1, M_2 and M_3 satisfy (i) of Definition 6.3. To verify condition (ii), notice that

$$I(M_1) = \{0\}, \qquad I(M_2) = \{0\} \times \mathbb{Z} \times \mathbb{Z} \qquad \text{and} \qquad I(M_3) = \{0\}\,.$$

Obvious substitutions of $\sqrt{2}$ by other roots and of the rotation by $\pi/4$ by other rotations give many more examples of matrices of this type.

Now we want to write down the form that equation (5.19) assumes for the affine system $\mathcal{F}_A(\Psi)$ when the dilation matrix is of Type-III. Besides (6.1) and (6.2), which correspond to the cases $m = 0$ and $m \in I(B) \setminus \{0\}$, we are led to consider the case $m \in I(B^\delta) \setminus I(B)$ (it is easy to see that this set is non empty for the matrix M_3). The details can be seen in the proof of Proposition 6.5 below. Before we present this proposition, we state the following lemma, which shows that, for matrices of Type-III, condition (ii) is true for any integer that is not divisible by δ.

Lemma 6.4. *Let M be a matrix of Type-III. If $\mathbb{Z}^n \cap M^s(\mathbb{Z}^n) \supsetneq I(M)$, then δ divides s.*

Proof. If $s = 0$ the result is obviously true. If $s < 0$, from $\mathbb{Z}^n \cap M^s(\mathbb{Z}^n) \supsetneq I(M)$ we deduce $M^{-s}\mathbb{Z}^n \cap \mathbb{Z}^n \supsetneq M^{-s}(I(M)) = I(M)$. Hence, without loss of generality we can assume $s > 0$. Write $s = c\delta + r$, $0 \le r < \delta$, $r \in \mathbb{Z}$, with c a non-negative integer. Choose $m \in \mathbb{Z}^n \cap M^s(\mathbb{Z}^n)$ and $m \notin I(M)$. Using (i) of Definition 6.3 we obtain

$$m \in M^s(\mathbb{Z}^n) = M^r M^{c\delta}(\mathbb{Z}^n) \subset M^r(\mathbb{Z}^n)\,.$$

Hence $m \in \mathbb{Z}^n \cap M^r(\mathbb{Z}^n)$. By (ii) of Definition 6.3, $m \in I(M)$ if $0 < r < \delta$, contrary to our assumption. We deduce that r must be zero, showing that s has to be a multiple of δ. □

Proposition 6.5. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that $B = A^t$ is a matrix of Type-III which is expanding on a subspace of $\mathbb{R}^n$. Then the affine system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions hold: (6.1), (6.2),*

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} \hat{\psi}^\ell(B^{-j\delta}\xi)\,\overline{\hat{\psi}^\ell(B^{-j\delta}(\xi+m))} = 0 \qquad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{6.8}$$

and all $m \in I(B^\delta) \setminus I(B)$, and

$$\sum_{\ell=1}^{L}\sum_{j\ge 0} \hat{\psi}^\ell(B^{j\delta}\xi)\,\overline{\hat{\psi}^\ell(B^{j\delta}(\xi+q))} = 0 \qquad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{6.9}$$

and all $q \in \mathbb{Z}^n \setminus B^\delta(\mathbb{Z}^n)$ (observe that $q \notin I(B^\delta)$).

Proof. The set $\mathbb{Z}^n$ is the disjoint union of the sets

$$\{0\}, \quad I(B)\setminus\{0\}, \quad I(B^\delta)\setminus I(B), \quad \text{and} \quad \mathbb{Z}^n \setminus I(B^\delta).$$

For $m \in I(B)$, condition (5.19) is equivalent to (6.1) if $m = 0$, and to (6.2) if $m \ne 0$.

Consider now

$$m \in I(B^\delta) \setminus I(B). \tag{6.10}$$

We claim that, for m as above, we have:

$$\mathcal{P}_m = \{j \in \mathbb{Z} : B^{-j}m \in \mathbb{Z}^n\} = \{i\delta : i \in \mathbb{Z}\}. \tag{6.11}$$

Since, for all $i \in \mathbb{Z}, B^{-i\delta}m \in B^{-i\delta}(I(B^\delta)) = I(B^\delta)$, and $I(B^\delta) \subset \mathbb{Z}^n$, it is clear that the set in the right hand side of (6.11) is contained in $\mathcal{P}_\alpha$m Suppose now that $j \in \mathcal{P}_m$ and $j \ne i\delta$ for each $i \in \mathbb{Z}^n$. We can then write $j = c\delta - s$ with $0 < s < \delta$. Therefore,

$$B^{-j}m = B^s B^{-c\delta}m \in B^s(B^{-c\delta}(I(B^\delta)) = B^s(I(B^\delta)) \subset B^s(\mathbb{Z}^n).$$

Also, $B^{-j}m \in \mathbb{Z}^n$ since $j \in \mathcal{P}_m$. Thus, $B^{-j}m \in \mathbb{Z}^n \cap B^s(\mathbb{Z}^n)$. Since B is of Type-III, by (ii) of Definition 6.3, $B^{-j}m \in I(B)$, and, consequently, $m \in B^j(I(B)) = I(B)$, contradicting the choice of m. This establishes (6.11).

For m as in (6.10), the equality (6.11) shows that (5.19) in Theorem 5.7 is equivalent to

$$\sum_{\ell=1}^{L}\sum_{i\in\mathbb{Z}}\hat{\psi}^{\ell}(B^{-i\delta}\xi)\,\overline{\hat{\psi}^{\ell}(B^{-i\delta}(\xi+m))}=0 \quad \text{for a.e. } \xi\in\mathbb{R}^n,$$

which is (6.8).

Choose, finally,

$$m\in\mathbb{Z}^n\setminus I(B^{\delta}). \tag{6.12}$$

By Lemma 6.2 applied to $M = B^{\delta}$, we deduce the existence of unique $d \in Z^{+}\cup\{0\}$ and $q \in \mathbb{Z}^n \setminus B^{\delta}(\mathbb{Z}^n)$ such that $m = B^{d\delta}q$. We claim that, for m as in (6.12), we have

$$\mathcal{P}_m=\{j\in\mathbb{Z}: B^{-j}\alpha\in\mathbb{Z}^n\}=\{k\delta: k\in\mathbb{Z},\, k\le d\}. \tag{6.13}$$

If $j = k\delta$ with $k \in \mathbb{Z}$, $k \le d$, then we can use (i) of Definition 6.3 to obtain $B^{-j}m = B^{-k\delta}B^{d\delta}q = B^{(d-k)\delta}q \in \mathbb{Z}^n$, since $d-k \ge 0$ and $q \in \mathbb{Z}^n$. This shows that the set on the right side of (6.13) is contained in $\mathcal{P}_m$. Choose now $j \in \mathcal{P}_m$ so that $B^{-j}m = s \in \mathbb{Z}^n$. Then, $s = B^{-j}m = B^{-j+d\delta}q$. Hence, $q = B^{j-d\delta}s \in \mathbb{Z}^n \cap B^{j-d\delta}(\mathbb{Z}^n)$. Also, $q \notin B^{\delta}(\mathbb{Z}^n)$, which implies $q \notin I(B)$. By Lemma 6.4 applied to $M = B$, we deduce $j = k\delta$ for some $k \in \mathbb{Z}$. Then, $q = B^{(k-d)\delta}s$ and $k-d \le 0$ (otherwise, if $k-d = t > 0$, then $q = B^{\delta t}s \in B^{\delta t}(\mathbb{Z}^n) \subset B^{\delta}(\mathbb{Z}^n)$ by (i) of Definition 6.3). This establishes (6.13).

For m as in (6.12), the equality (6.13) shows that (5.19) in Theorem 5.7 is equivalent to

$$\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z},\,k\le d}\hat{\psi}^{\ell}(B^{-k\delta}\xi)\,\overline{\hat{\psi}^{\ell}(B^{-k\delta}(\xi+B^{d\delta}q))}=0 \quad \text{for a.e. } \xi\in\mathbb{R}^n. \tag{6.14}$$

The change of variables $\xi = B^{d\delta}\eta$ shows that (6.14) is equivalent to

$$\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z},\,k\le d}\hat{\psi}^{\ell}(B^{(d-k)\delta}\eta)\,\overline{\hat{\psi}^{\ell}(B^{(d-k)\delta}(\eta+q))}=0 \quad \text{for a.e. } \eta\in\mathbb{R}^n. \tag{6.15}$$

Finally, the change of indices $j = d-k$ in the summation shows that (6.15) is equivalent to (5.19) in Theorem 5.7 for the values of m given by (6.12). This finishes the proof of the Proposition. □

Example 9. For the matrix $A = \begin{pmatrix}\sqrt{2} & 0\\ 0 & 1\end{pmatrix}$ and a single $\psi \in L^2(\mathbb{R}^2)$, it follows from Proposition 6.5 that the affine system $\mathcal{F}_A(\psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^2)$ if and only if

$$\sum_{j\in\mathbb{Z}}|\hat{\psi}(2^{j/2}\xi_1,\ \xi_2)|^2=1 \quad \text{for a.e. } \xi_1,\xi_2\in\mathbb{R},$$

$$\sum_{j\in\mathbb{Z}}\hat{\psi}(2^{j/2}\xi_1,\ \xi_2)\,\overline{\hat{\psi}(2^{j/2}\xi_1,\ \xi_2+m_2)}=0 \quad \text{for a.e. } \xi_1,\xi_2\in\mathbb{R},\ \text{and all } m_2\in\mathbb{Z}\setminus\{0\},$$

$$\sum_{j\ge 0}\hat{\psi}(2^{j}\xi_1,\ \xi_2)\,\overline{\hat{\psi}(2^{j}(\xi_1+q_1),\ \xi_2+m_2)}=0 \quad \text{for a.e. } \xi_1,\xi_2\in\mathbb{R},\ \text{and all } q_1\in\mathbb{Z}\setminus 2\mathbb{Z}, m_2\in\mathbb{Z}.$$

6.4 Matrices expanding on $\mathbb{R}^n$

If the dilation matrix $A \in GL_n(\mathbb{R})$ is expanding (i.e., it is expanding on $F = \mathbb{R}^n$), then the form of the equalities (5.19) can be expressed in a way that is yet more similar to the "classical" equalities (1.6) and (1.7). That is, one equality, corresponding to the case $q = 0$ in (6.16), is the Calderón condition, while the others have a form that is a direct generalization of (1.7).

Theorem 6.6. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, $B = A^t$ and $A \in GL_n(\mathbb{R})$ be expanding on $\mathbb{R}^n$. Then the system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions holds: (6.1) and*

$$\sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_q} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + q))} = \delta_{q,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{6.16}$$

and all $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$, where $\mathcal{P}_q = \{j \in \mathbb{Z} : B^{-j}q \in \mathbb{Z}^n\}$.

Proof. We must show the equivalence of (5.19) and (6.16) when B is expanding.
Let us first observe that, in this case,

$$I^+(B) \;=\; \bigcap_{j \geq 0} B^j(\mathbb{Z}^n) \;=\; \{0\}.$$

This is an immediate consequence of inequality (5.6) in Lemma 5.2. Indeed, if $x \in I^+(B)$, then, for each $j \geq 0$, there exists $m_j \in \mathbb{Z}^n$ such that $x = B^j\, m_j$. By (5.6), we then have $|m_j| = |B^{-j}\, x| \leq \frac{1}{k}\, \gamma^{-j}\, |x|$ and the last expression tends to zero as $j \to \infty$ since $\gamma > 1$. Hence, m_j must be zero since the last expression must be strictly smaller than 1, the minimal norm for a non-zero lattice point, for j large enough.

It is clear that (5.19) implies (6.16), and that the two expressions are the same when $q = m = 0$, which also shows that (5.19) implies (6.1). Therefore, we only have to show that equality (5.19), for $m \in \mathbb{Z}^n \setminus \{0\}$, is equivalent to one of the equalities (6.16), for an appropriate $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$. We first claim that any such $m \in \mathbb{Z}^n \setminus \{0\}$ can be written as $m = B^d\, q$ for some $d \geq 0$ and $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$, provided B is expanding. To prove this claim proceed as follows. If $m \notin B(\mathbb{Z}^n)$, then set $d = 0$ and $q = m$; while, if $m \in B(\mathbb{Z}^n)$, then set $m = B\, m_1$ and we reason for m_1 as we just did for m: either $m_1 \notin B(\mathbb{Z}^n)$, and we are done with $q = m_1$, or $m_1 \in B(\mathbb{Z}^n)$ in which case $m = B\, m_1 = B^2\, m_2$ with $m_2 \in \mathbb{Z}^n$. This process must stop after a finite number of steps; otherwise, $m = B^j\, m_j$ for a $m_j \in \mathbb{Z}^n$ for all $j \geq 0$. This would imply that $m \in I^+(B) = \cap_{j \geq 0} B^j(\mathbb{Z}^n)$ and we reach a contradiction. This establishes the last claim.

Thus, if (5.19) with $m \neq 0$ is true, then we can write $m = B^d\, q$ for some $d \geq 0$ and $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$. Now, using the change of variables $\xi = B^d \eta$ and the fact that $j \in \mathcal{P}_{B^d q}$ if and only if $k = j - d \in \mathcal{P}_q$, we obtain

$$\sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_m} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + m))} = \sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_{B^d q}} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + B^d\, q))}$$

$$= \sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_{B^d q}} \hat{\psi}^\ell(B^{-j+d}\eta)\, \overline{\hat{\psi}^\ell(B^{-j+d}(\eta + q))} = \sum_{\ell=1}^{L} \sum_{k \in \mathcal{P}_q} \hat{\psi}^\ell(B^{-k}\eta)\, \overline{\hat{\psi}^\ell(B^{-k}(\eta + q))}. \quad \square$$

Remark. The types of matrices we have considered in this section do not cover all the possible matrices that are expanding on subspaces of $\mathbb{R}^n$. For example, the matrix

$$M = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

does not belong to any of the above types. It is a significant problem to understand the form that (5.8) in Theorem 5.3 assumes for all the dilation matrices expanding on subspaces of $\mathbb{R}^n$, in order to obtain expressions that do not involve the sets Λ and $\mathcal{P}_\alpha$, in the same spirit as done in Propositions 6.1, 6.3, and 6.5. The problem has been completely solved in [8] for dimension 1 (the matrix is a real number a, with $a > 1$), where they have considered the sets

$$\begin{aligned} E_1 &= \{a \in \mathbb{R} : a > 1, \text{ and } a^j \in \mathbb{Z} \text{ for some integer } j > 0\}, \\ E_2 &= \{a \in \mathbb{R} : a > 1, \text{ and } a^j \in \mathbb{Q} \setminus \mathbb{Z} \text{ for some integer } j > 0\}, \\ E_3 &= \{a \in \mathbb{R} : a > 1, \text{ and } a^j \notin \mathbb{Q} \text{ for all integer } j > 0\}, \end{aligned}$$

and they have given simpler expressions for the characterization equations in each one of these cases. The general problem in dimension $n > 1$ remains open.

7 Quasi-affine systems

Let $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$, and $A \in GL_n(\mathbb{R})$. The **quasi-affine system** generated by Ψ, denoted as $\widetilde{\mathcal{F}}_A(\Psi)$, is defined by

$$\widetilde{\mathcal{F}}_A(\Psi) = \{\widetilde{\psi}^\ell_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \dots, L\}, \tag{7.1}$$

where

$$\widetilde{\psi}^\ell_{j,k} = \begin{cases} |\det A|^{\frac{j}{2}}\, T_k\, D_{A^j}\, \psi^\ell, & j < 0 \\ D_{A^j}\, T_k\, \psi^\ell, & j \geq 0. \end{cases}$$

The notion of quasi-affine system was introduced by A. Ron and Z. Shen in [28] under the assumption that $A \in GL_n(\mathbb{Z})$. It is easy to verify that when the dilation matrix A preserves the integer lattice (i.e., $A\,\mathbb{Z}^n \subset \mathbb{Z}^n$), then the quasi-affine systems $\widetilde{\mathcal{F}}_A$, unlike the affine systems $\mathcal{F}_A$, are invariant under integer translations. Ron and Shen discovered that there is some sort of equivalence between the affine systems $\mathcal{F}_A(\Psi)$ and the corresponding quasi-affine systems $\widetilde{\mathcal{F}}_A(\Psi)$. In particular, they obtained the following result (discovered in [28] under a mild decay assumption on ψ, and proved in full generality in [9]):

Theorem 7.1 ([9]). *Let $A \in GL_n(\mathbb{Z})$ be expanding. Then the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$ is a normalized tight frame if and only the corresponding affine system $\mathcal{F}_A(\Psi)$ is a normalized tight frame.*

More general notions of equivalence are also proved in [9], such as the fact that affine and quasi-affine frames are equivalent. It follows from Theorem 7.1 that, once the quasi-affine systems $\widetilde{\mathcal{F}}_A(\Psi)$ have been studied using techniques from the theory of shift-invariant spaces, then the results can be transferred to the corresponding affine systems $\mathcal{F}_A(\Psi)$ (cf. [29, 3, 21] for an application of this approach to the characterization of affine tight and dual frames).

The main result of this section is yet another application of Theorem 2.1, which gives the following characterization of normalized quasi-affine tight frames.

Theorem 7.2. *Let $\Psi = \{\psi^1, \cdots \psi^L\} \subset L^2(\mathbb{R}^n)$ and let $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. Then the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$, given by (7.1), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}^-\cup\mathcal{Q}_m} \hat{\psi}^\ell(B^{-j}\xi)\,\overline{\hat{\psi}^\ell(B^{-j}(\xi+m))} = \delta_{m,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{7.2}$$

for all $m \in \mathbb{Z}^n$, and

$$\sum_{\ell=1}^{L} \sum_{j\in\mathcal{Q}_\alpha} \hat{\psi}^\ell(B^{-j}\xi)\,\overline{\hat{\psi}^\ell(B^{-j}(\xi+\alpha))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{7.3}$$

for all $\alpha \in \Lambda^q \setminus \mathbb{Z}^n$, where $\Lambda^q = \bigcup_{j\in\mathbb{Z}^+\cup\{0\}} B^j(\mathbb{Z}^n)$ and $\mathcal{Q}_x = \{j \in \mathbb{Z}^+ \cup \{0\} : B^{-j}x \in \mathbb{Z}^n\}$.

Remarks.

1. Our result is not restricted to integer-valued expanding matrices, as is the classical result of A. Ron and Z. Shen, but is valid for real matrices expanding on subspaces, as defined in Section 5. As a corollary to our result, we will show the equivalence of affine and quasi-affine systems may not hold if the matrix is not integer-valued (see Example 11).

2. The expressions (7.2) and (7.3) share some features with equation (5.7) in Theorem 5.4. Observe that, if $A \in GL_n(\mathbb{Z})$ (hence, $A\mathbb{Z}^n \subset \mathbb{Z}^n$), then $\Lambda^q = \mathbb{Z}^n$ and equation (7.3) is void. We will show in Proposition 8.2 that, in the case of integer-valued matrices, the expressions (7.2) and (5.7) are equivalent, and this gives a new proof of Theorem 7.1. This provides a new simple proof of Theorem 7.1 in the case of normalized tight frames. Additionally, our result holds for matrices expanding on subspaces (while the result in [28], [9] requires expanding matrices).

3. If $\alpha = 0$, then $\mathbb{Z}^- \cup \mathcal{Q}_0 = \mathbb{Z}$, and (7.2) becomes the Calderón condition

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 = 1, \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

exactly as in the case of affine systems.

Proof of Theorem 7.2. Apply Theorem 2.1 with

$$\mathcal{P} = \big\{(j,\ell) : j \in \mathbb{Z},\, \ell = 1, 2, \dots, L\big\},$$

$$g_p \equiv g_{(j,l)} = \begin{cases} |\det A|^{\frac{j}{2}} \, D_{A^j}\, \psi^\ell, & j \,<\, 0 \\ D_{A^j}\, \psi^\ell, & j \,\geq\, 0, \end{cases} \qquad C_p \equiv C_{(j,\ell)} = \begin{cases} I, & j \,<\, 0,\, \ell = 1, \ldots, L \\ A^{-j}, & j \,\geq\, 0,\, \ell = 1, \ldots, L. \end{cases}$$

With this choice for $\mathcal{P}$, g_p and C_p, using the relation $T_{A^{-j}k}\, D_{A^j}\, \psi = D_{A^j}\, T_k\, \psi$, it follows that the system $\{T_{C_p\, k}\, g_p : \, k \in \mathbb{Z}^n, p \in \mathcal{P}\}$

$$\{T_{C_p\, k}\, g_p : \, k \in \mathbb{Z}^n, p \in \mathcal{P}\} = \begin{cases} \{|\det A|^{j/2}\, T_k\, D_{A^j}\, \psi^\ell : \, k \in \mathbb{Z}^n, j < 0, \ell = 1, \cdots, L\} \\ \{T_{A^{-j}\,k}\, D_{A^j}\, \psi^\ell : \, k \in \mathbb{Z}^n, j \geq 0, \ell = 1, \cdots, L\} \end{cases}$$

$$= \begin{cases} \{|\det A|^{j/2}\, T_k\, D_{A^j}\, \psi^\ell : \, k \in \mathbb{Z}^n, j < 0, \ell = 1, \cdots, L\} \\ \{D_{A^j}\, T_k\, \psi^\ell : \, k \in \mathbb{Z}^n, j \geq 0, \ell = 1, \cdots, L\} \end{cases}$$

is the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$.

With the same choices, the set Λ, given by (2.2), is

$$\Lambda \,=\, \Lambda^q \,=\, (\bigcup_{j\in\mathbb{Z}^-} I^j\, \mathbb{Z}^n) \cup (\bigcup_{j\in\mathbb{Z}^+\cup\{0\}} B^j\, \mathbb{Z}^n) \,=\, \bigcup_{j\in\mathbb{Z}^+\cup\{0\}} B^j\, \mathbb{Z}^n,$$

and the set $\mathcal{P}_\alpha$, given by (2.3), is

$$\begin{aligned} \mathcal{P}_\alpha =& \{(j,\ell) : j \in \mathbb{Z}^-,\, \ell = 1, \ldots, L : \, \alpha \in \mathbb{Z}^n\} \,\cup \\ & \cup \{(j,\ell) : j \in \mathbb{Z}^+ \cup \{0\},\, \ell = 1, \ldots, L : \, B^{-j}\alpha \in \mathbb{Z}^n\}. \end{aligned}$$

Thus, if $\alpha = m \in \mathbb{Z}^n$, then $\mathcal{P}_\alpha = (\mathbb{Z}^- \cup \mathcal{Q}_m) \times \{1, \ldots, L\}$, and equation (7.2) follows from (2.6) in Theorem 2.1. Similarly, if $\alpha \in \Lambda^q \setminus \mathbb{Z}^n$, then $\mathcal{P}_\alpha = \mathcal{Q}_\alpha \times \{1, \ldots, L\}$, and equation (7.3) follows from (2.6) in Theorem 2.1.

Therefore, all that is left to prove is that the hypothesis (2.6) is satisfied in this particular case. Choose $f \in \mathcal{D}_E$, where $\mathcal{D}_E$ is a dense subspace of $L^2(\mathbb{R}^n)$ defined by (5.10), and E is a complementary subspace to F as in Definition 5.1. Thus, we need to show that $L^q(f) < \infty$ for $f \in \mathcal{D}_E$, where

$$\begin{aligned} L^q(f) =& \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}^-} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + m)|^2\, |\det A^j|\, |(D_A^j \psi^\ell)^\wedge(\xi)|^2\, d\xi \,+ \\ &+ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}^+\cup\{0\}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + B^j m)|^2\, |\det A^j|\, |(D_A^j \psi^\ell)^\wedge(\xi)|^2\, d\xi. \end{aligned}$$

Since $(D_A^j \psi)^\wedge(\xi) = |\det A|^{-1/2}\, \hat{\psi}(B^{-j}\xi)$, then

$$\begin{aligned} L^q(f) =& \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}^-} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + m)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi \,+ \\ &+ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}^+\cup\{0\}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + B^j m)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi. \end{aligned}$$

Write $L^q(f) = L^q_-(f) + L^q_+(f)$, where $L^q_-(f)$ and $L^q_+(f)$ denote the sums corresponding to $j \in \mathbb{Z}^-$ and $j \in \mathbb{Z}^+ \cup \{0\}$, respectively.

Consider first the expression for $L^q_-(f)$. Since $f \in \mathcal{D}_E$, there exists an $R > 0$ such that $\operatorname{supp} \hat{f} \subset B(R)$. In order to have $L^q(f) \neq 0$, we must have $|\xi| \leq R$ and $|\xi + m| \leq R$. Therefore, $|m| \leq 2\,R$, and the sum with respect to m in $L^q_-(f)$ is finite, where the number of $m \in \mathbb{Z}^n$ is at most $(2\,R)^n$. Furthermore, if the quasi-affine system $\widetilde{\mathcal{F}_A}(\Psi)$ is a normalized tight frame for $L^2(\mathbb{R}^n)$, then, by Proposition 4.1 applied to the quasi-affine system (7.1), we deduce that

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 \leq 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n\,.$$

This inequality also holds if we assume (7.3) (take $\alpha = 0$). Together with the bound for the sum with respect to m, the last inequality shows that:

$$L^q_-(f) \leq (2\,R)^n\,|B(R)|\,\|\hat{f}\|^2_\infty. \tag{7.4}$$

Finally, consider the expression for $L^q_+(f)$. It is clear that

$$L^q_+(f) \leq L(f) = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + B^j m)|^2\,|\hat{\psi}^\ell(B^{-j}\xi)|^2\,d\xi,$$

and $L(f) < \infty$, by Proposition 5.6. □

A simple application of Theorem 4.2 to the quasi-affine systems $\widetilde{\mathcal{F}}_A$ yields another characterization of quasi-affine normalized tight frames.

Theorem 7.3. *Let $\Psi = \{\psi^1, \cdots \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. Then the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$, given by (7.1), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if it is a Bessel system with constant 1 and the Calderón condition (6.1) holds.*

Proof. Apply Theorem 4.2 with $\mathcal{P}$, g_p and C_p as in the proof of Theorem 7.2. The fact that condition (2.6) is satisfied for all $f \in \mathcal{D}_E$, where $\mathcal{D}_E$ is a dense subspace of $L^2(\mathbb{R}^n)$ defined by (5.10), and E is a complementary subspace to F as in Definition 5.1, follows from the same argument as in the proof of Theorem 7.2. □

Using Theorem 7.3, we make the following observation.

Corollary 7.4. *Let $\Psi = \{\psi^1, \cdots \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. If the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$ is a normalized tight frame for $L^2(\mathbb{R}^n)$, then the corresponding affine system $\mathcal{F}_A(\Psi)$ is also a normalized tight frame for $L^2(\mathbb{R}^n)$.*

In order to prove Corollary 7.4, we need the following Lemma, which is adapted from [9, Theorem 2].

Lemma 7.5. *Let $\Psi = \{\psi^1, \cdots \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$. If the system $\mathcal{F}^+_A(\Psi) = \{D_{A^j}\,T_k\,\psi^\ell : j \in \mathbb{Z}^+ \cup \{0\}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}$ is a Bessel system with constant B, then the affine system $\mathcal{F}_A(\Psi)$, given by (5.1), has the same property.*

Proof. Since $\mathcal{F}_A^+(\Psi)$ is a Bessel system with constant B, then

$$\sum_{\ell=1}^{L} \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} |\langle f, D_A^j T_k \psi^\ell \rangle|^2 \leq B \|f\|_2^2,$$

for all $f \in L^2(\mathbb{R}^n)$. Thus, given $N \in \mathbb{N}$ and any $g \in L^2(\mathbb{R}^n)$, from the last inequality with $f = D_A^N g$ we deduce that

$$\sum_{\ell=1}^{L} \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} |\langle D_A^N g, D_A^j T_k \psi^\ell \rangle|^2 \leq B \|D_A^N g\|_2^2 = B \|g\|_2^2, \tag{7.5}$$

for all $g \in L^2(\mathbb{R}^n)$ and $N \in \mathbb{N}$. Since $\langle D_A^N g, D_A^j T_k \psi^\ell \rangle = \langle g, D_A^{j-N} T_k \psi^\ell \rangle$, then from (7.5) we have that

$$\sum_{\ell=1}^{L} \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} |\langle g, D_A^{j-N} T_k \psi^\ell \rangle|^2 \leq B \|g\|_2^2,$$

for all $g \in L^2(\mathbb{R}^n)$ and $N \in \mathbb{N}$. Thus, applying the change of indices $i = j - N$ we obtain

$$\sum_{\ell=1}^{L} \sum_{i \geq -N} \sum_{k \in \mathbb{Z}^n} |\langle g, D_A^{i} T_k \psi^\ell \rangle|^2 \leq B \|g\|_2^2,$$

and the result then follows by taking the limit for N approaching infinity. □

Proof of Corollary 7.4. If the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$, given by (7.1), is a Bessel system with constant 1, then so is the system $\mathcal{F}_A^+(\Psi) = \{D_{A^j} T_k \psi^\ell : j \in \mathbb{Z}^+ \cup \{0\}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}$, and so, by Lemma 7.5, is the corresponding affine system $\mathcal{F}_A(\Psi)$. By the item **3** of the Remarks after Theorem 7.2, the systems $\widetilde{\mathcal{F}}_A(\Psi)$ and $\mathcal{F}_A(\Psi)$ satisfy the same Calderón condition, and this completes the proof. □

8 Quasi-affine systems: special dilation matrices

In this section, we are going to analyze, in the same spirit as in Section 6, the forms that the characterization equations (7.2) and (7.3) assume corresponding to different values of $m \in \mathbb{Z}^n$ and $\alpha \in \Lambda^q$. These differences will depend on the dilation matrix A, expanding on subspaces of $\mathbb{R}^n$, similarly to the situation we encountered in Section 6. We shall examine matrices of Type-I and Type-II, as defined in Section 6.

As a consequence of the results we discuss in this Section, we have that for $A \in GL_n(\mathbb{R})$ of Type-II (i.e., A has integer entries), the affine system $\mathcal{F}_A(\Psi)$ is a normalized tight frame if and only if the corresponding quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$ **has** the same property (see Theorem 7.1 and the references given before its statement for this equivalence in the case of expanding dilation matrices in $GL_n(\mathbb{R})$). On the other hand, we give examples of matrices A of Type-I for which $\mathcal{F}_A(\Psi)$ is a normalized tight frame, but the corresponding quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$ **does not have** the same property.

If we take $m = 0$ in (7.2), we have $\mathcal{Q}_m = \mathbb{Z}^+ \cup \{0\}$ (the set $\mathcal{Q}_m$ is defined in Theorem 7.2). Then (7.2) is the Calderón condition

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 = 1 \quad \text{for a. e.} \quad \xi \in \mathbb{R}^n\,. \tag{8.1}$$

If $m \neq 0$ and $m \in I^q(B) = \bigcap_{\{i\in\mathbb{Z},\ i\geq 0\}} B^i(\mathbb{Z}^n)$, we have $\mathcal{Q}_m = \mathbb{Z}^+ \cup \{0\}$, and (7.2) is equivalent to

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi+m))} = 0 \quad \text{for a. e.} \quad \xi \in \mathbb{R}^n\,. \tag{8.2}$$

The set $I^q(B)$ is contained in $\mathbb{Z}^n$. If B is an expanding matrix on $\mathbb{R}^n$ we have $I^q(B) = \{0\}$ and, consequently, condition (8.2) is not present. For the matrices of Example 2 in Section 5, we have $I^q(B) = \{0\} \times \mathbb{Z}$. For the matrices of Example 3 in Section 5, we have $I^q(B) = E \cap \mathbb{Z}^2$. For the matrices of Example 4 in Section 5, the set $I^q(B)$ depends on the angle of rotation θ. For the matrices of Example 5 in Section 5, we have $I^q(B) = \{0\} \times \mathbb{Z} \times \mathbb{Z}$ when b is an integer.

For other values of $\alpha \in \Lambda$, the form that (7.2) and (7.3) assume depends on the dilation matrix $B = A^t \in GL_n(R)$. As special applications of Theorem 7.2 we treat below matrices of Type-I and Type-II, as defined in Section 6. In order to avoid excessive technical discussions, we leave to the reader the exploration of how dilation matrices of Type-III are involved in the quasi-affine case.

8.1 Matrices of Type-I

Recall that $B \in GL_n(\mathbb{R})$ is a matrix of **Type-I** if $B^j(\mathbb{Z}^n) \cap \mathbb{Z}^n = I(B) = \bigcap_{i\in\mathbb{Z}} B^i(\mathbb{Z}^n)$ for all $j \in \mathbb{Z}\backslash\{0\}$, according to Definition 6.1. Then, in this situation, $I^q(B) \subset B(\mathbb{Z}^n)\cap\mathbb{Z}^n = I(B) \subset I^q(B)$, so that we have

$$I^q(B) = I(B). \tag{8.3}$$

In view of this equality, condition (6.2) for affine systems and condition (8.2) for quasi-affine systems range over the same values of m.

We can now give the form that the equations that appear in Theorem 7.2 assume in this situation.

Proposition 8.1. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that $B = A^t$ is a matrix of Type-I which is expanding on a subspace of $\mathbb{R}^n$. Then the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$, given by (7.1), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions hold: the Calderón condition (8.1), (8.2),*

$$\sum_{\ell=1}^{L} \hat{\psi}^\ell(\xi)\, \overline{\hat{\psi}^\ell(\xi+m)} = 0 \qquad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{8.4}$$

and all $m \in \mathbb{Z}^n \setminus I^q(B)$, and

$$\sum_{\ell=1}^{L} \sum_{j\geq 1} \hat{\psi}^\ell(B^j\xi)\, \overline{\hat{\psi}^\ell(B^j(\xi+m))} = 0 \qquad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{8.5}$$

and all $m \in \mathbb{Z}^n \setminus I^q(B)$.

Proof. We apply Theorem 7.2. We have already observed that (8.1) and (8.2) are the cases $m = 0$ and $m \in I^q(B) \setminus \{0\}$ of (7.2). We now need to consider the cases of $m \in \mathbb{Z}^n \setminus I^q(B)$ and $\alpha \in \Lambda^q \setminus I^q(B)$.
For $m \in \mathbb{Z}^n \setminus I^q(B)$, since B is of Type-I, we obtain $\mathcal{Q}_m = \{0\}$. In this case, (7.2) is equivalent to

$$\sum_{\ell=1}^{L} \sum_{j \le 0} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + m))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{8.6}$$

For $\alpha \in \Lambda^q \setminus \mathbb{Z}^n$, write $\alpha = B^{j_0}m$ for some $j_0 \in \mathbb{Z}^+$ and $m \in \mathbb{Z}^n \setminus I^q(B)$. Then, $j \in \mathcal{Q}_\alpha$ if and only if $j \in \mathbb{Z}^+ \cup \{0\}$ and $B^{-j}\alpha = B^{-j+j_0}m \in \mathbb{Z}^n$. Since B is of Type-I, we deduce that $j = j_0$, so that $\mathcal{Q}_\alpha = \{j_0\}$. In this case,

$$\sum_{\ell=1}^{L} \hat{\psi}^\ell(B^{-j_0}\xi)\, \overline{\hat{\psi}^\ell(B^{-j_0}(\xi + B^{j_0}m))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Change $B^{-j_0}\xi$ to η to obtain (8.4). Finally, observe that the terms with $j = 0$ in (8.6) add up to zero by the just proved equation (8.4), so that (8.6) becomes (8.5) after changing j to $-j$, □

Example 10. If $a \in \mathbb{R},\ a > 1$, with $a^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z}^+$, and a single $\psi \in L^2(\mathbb{R})$, it follows from Proposition 8.1 that the quasi-affine system $\widetilde{\mathcal{F}}_A(\psi)$, given by (7.1), is a normalized tight frame for $L^2(\mathbb{R})$ if and only if

$$\sum_{j \in \mathbb{Z}} \mid \hat{\psi}(a^j\xi) \mid^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}, \tag{8.7}$$

$$\hat{\psi}(\xi)\overline{\hat{\psi}(\xi + m)} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}, \text{ and all } m \in \mathbb{Z} \setminus \{0\}, \tag{8.8}$$

and

$$\sum_{j \ge 1} \hat{\psi}(a^j\xi)\overline{\hat{\psi}(a^j(\xi + m))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}, \text{ and all } m \in \mathbb{Z} \setminus \{0\}. \tag{8.9}$$

Remark. Comparing Propositions 6.1 and 8.1, it is clear that, for matrices of Type-I expanding on subspaces of $\mathbb{R}^n$, if the quasi-affine frame is a normalized tight frame for $L^2(\mathbb{R}^n)$, then also the corresponding affine frame is a normalized tight frame for $L^2(\mathbb{R}^n)$. Of course, this is in agreement with Corollary 7.4. On the other hand, (8.5) does not appear in Proposition 6.1: this fact will allow us to exhibit affine normalized tight frames for $L^2(\mathbb{R}^n)$, for which the corresponding quasi-affine system is not a normalized tight frame for $L^2(\mathbb{R}^n)$. This is presented in the following example.

Example 11. We carry out the construction in dimension $n = 1$. Let $a \in \mathbb{R}$, $a > 1$, and $a \notin \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $r - 1 < a < r$. Assume $r - (1/2) \le a < r$ (the case $r - 1 < a < r - (1/2)$

requires only minor modifications from the example we present below). Choose $\beta \in \mathbb{R}$ such that $r-1 < a < \beta < r$, and let $\epsilon = r - \beta > 0$.

Write $J = (r - \frac{1}{2}, \beta)$, and $I = (\beta, a(r - \frac{1}{2})$, so that $J \cup I = (r - \frac{1}{2}, a(r - \frac{1}{2}))$. Observe that $a(r - \frac{1}{2}) > (r - \frac{1}{2})^2 = r^2 - r + \frac{1}{4} > \beta$, since $r \geq 2$. Choose j_0 to be a positive integer large enough so that

$$\frac{a(r - \frac{1}{2})}{a^{j_0}} < \min\{\epsilon, \beta - a\}. \tag{8.10}$$

Let $K = \frac{1}{a^{j_0}} I$, and consider $W_a = (\pm K) \cup (\pm J)$. Define ψ_a by

$$\hat{\psi}_a = \chi_{W_a}.$$

Since $a^{j_0} K \cup J = (\pm I) \cup (\pm J) = \pm(r - \frac{1}{2}, a(r - \frac{1}{2}))$, (8.7) holds for ψ_a. By (8.10), $W_a \cap (W_a + m) = \emptyset$ for all $m \in \mathbb{Z} \setminus \{0\}$. Hence (8.8) is true.

If we choose $a \in \mathbb{R},\ a > 1$, such that $a^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z}^+$, from Example 10 we deduce that the affine system $\mathcal{F}_a(\psi_a)$ is a normalized tight frame $L^2(\mathbb{R}^n)$. Moreover, by (8.10), if $\xi \in a^{-1}K$,

$$\hat{\psi}_a(a\xi)\, \overline{\hat{\psi}_a(a\xi + a)} = 1, \qquad \text{(since } K + a \subset J\text{)},$$

and for $j \in \mathbb{Z}^+$, $\hat{\psi}_a(a^j\xi)\, \overline{\hat{\psi}_a(a^j\xi + a^j)} = 0$, since $\hat{\psi}_a(a^j\xi) = 0$. Thus, (8.9) does not hold for $m = 1$ and, consequently, the quasi-affine system $\widetilde{\mathcal{F}}_a(\psi_a)$ is **not** a normalized tight frame for $L^2(\mathbb{R})$.

8.2 Matrices of Type-II

Recall that $B \in GL_n(\mathbb{R})$ is a matrix of **Type-II** if $B(\mathbb{Z}^n) \subset \mathbb{Z}^n$, according to Definition 6.2. Then, in this situation,

$$\cdots \subset B^2(\mathbb{Z}^n) \subset B(\mathbb{Z}^n) \subset \mathbb{Z}^n \subset B^{-1}(\mathbb{Z}^n) \subset B^{-2}(\mathbb{Z}^n) \subset \cdots$$

and, consequently,

$$I^q(B) = \bigcap_{\{i \in \mathbb{Z},\, i \geq 0\}} B^i(\mathbb{Z}^n) = \bigcap_{i \in \mathbb{Z}} B^i(\mathbb{Z}^n) = I(B)\,. \tag{8.11}$$

In view of this equality, condition (6.2) for affine systems and condition (8.2) for quasi-affine systems range over the same values of m.

We can now give the form that the equations that appear in Theorem 7.2 assume in this situation.

Proposition 8.2. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that $B = A^t$ is a matrix of Type-II which is expanding on a subspace of $\mathbb{R}^n$. Then the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$, given by (7.1), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions hold: the Calderón condition (8.1), (8.2), and*

$$\sum_{\ell=1}^{L} \sum_{j \geq 0} \hat{\psi}^\ell(B^j\xi)\, \overline{\hat{\psi}^\ell(B^j(\xi + q))} = 0 \quad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{8.12}$$

and all $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$.

Proof. We apply Theorem 7.2. We have already observed that (8.1) and (8.2) are the cases $m = 0$ and $m \in I^q(B) \setminus \{0\}$ of (7.2).
For matrices of Type-II, $\Lambda^q = \bigcup_{j \in \mathbb{Z}^+ \cup \{0\}} B^j(\mathbb{Z}^n) = \mathbb{Z}^n$ by the inclusions that precede (8.11), and (7.3) is void. Thus, we only need to consider the case $m \in \mathbb{Z}^n \setminus I^q(B)$. Observe that, since $I^q(B) = I(B)$, by (8.11), we can apply Lemma 6.2 to deduce the existence of unique $d \in \mathbb{Z} + \cup \{0\}$ and $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$ such that $m = B^d q$.
We want to examine $\mathcal{Q}_m = \{j \in \mathbb{Z}^+ \cup \{0\} : B^{-j} m \in \mathbb{Z}^n\}$. If $j \in \mathcal{Q}_m$, then $B^{-j} B^d q = B^{-j} m \in \mathbb{Z}^n$. We must have $-j + d \geq 0$ (otherwise, with $-j + d = -\ell < 0$, we deduce $B^{-\ell} q \in \mathbb{Z}^n$, which implies $q \in B^\ell(\mathbb{Z}^n) \subset B(\mathbb{Z}^n)$, contrary to our choice of q).
Also, if $0 \leq j \leq d$, then $j \in \mathcal{Q}_m$; in fact, since $-j + d \geq 0$ and B is of Type-II, we obtain $B^{-j} m = B^{-j+d} q \in \mathbb{Z}^n$. Thus, $\mathcal{Q}_m = \{0, 1, 2, \cdots, d\}$, and (7.2) is equivalent to

$$\sum_{\ell=1}^{L} \sum_{j<0} \hat{\psi}^\ell(B^{-j}\xi) \, \overline{\hat{\psi}^\ell(B^{-j}(\xi + B^d q))} + \sum_{\ell=1}^{L} \sum_{j=0}^{d} \hat{\psi}^\ell(B^{-j}\xi) \, \overline{\hat{\psi}^\ell(B^{-j}(\xi + B^d q))} = 0$$

for a.e. $\xi \in \mathbb{R}^n$. Collecting the two sums with j ranging from $-\infty$ to d, doing the change of variables $\xi = B^d \eta$, and changing the index of summation to $k = -j + d$, we obtain (8.12). This finishes the proof of the Proposition. $\square$

Remarks.

(1) Comparing Propositions 6.3 and 8.2, and taking into account the equality (8.11), it is clear that for matrices of Type-II (i.e. matrices with integer entries), expanding on subspaces of $\mathbb{R}^n$, the quasi-affine frame is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the affine frame is a normalized tight frame for $L^2(\mathbb{R}^n)$. This generalizes Theorem 7.1 to the case of matrices which are not just expanding, but expanding on subspaces of $\mathbb{R}^n$.

(2) We have proved in Example 7 that the equivalence stated in the above remark does not carry over to matrices of Type-I. On the other hand, recently M. Bownik [4] has modified the quasi-affine system (7.1) to obtain this equivalence for rational dilation matrices expanding on $\mathbb{R}^n$ (that is, matrices with rational entries).

9 Dual systems

In this section, we consider the case of systems satisfying a reproducing formula of the form

$$v = \sum_{\alpha \in \mathcal{A}} \langle v, e_\alpha \rangle \, \eta_\alpha, \qquad v \in \mathcal{H}$$

where the "analyzing" family $\{\eta_\alpha\}_{\alpha \in \mathcal{A}}$ differs from the "synthetizing" family $\{e_\alpha\}_{\alpha \in \mathcal{A}}$. Since the results that we shall present in this section follow for the most part by simple adaptations of the arguments used in the tight frame case, in order to avoid repetitions, we will omit or simply sketch some of the proofs.

Let $e = \{e_\alpha\}_{\alpha \in \mathcal{A}}$ and $\eta = \{\eta_\alpha\}_{\alpha \in \mathcal{A}}$ be Bessel systems for $\mathcal{H}$. Then $\{\eta_\alpha\}_{\alpha \in \mathcal{A}}$ is called a **dual**

system to $\{e_\alpha\}_{\alpha\in\mathcal{A}}$, if

$$K_{e,\eta}(v,w) \;=\; \sum_{\alpha\in\mathcal{A}} \langle v, e_\alpha\rangle \,\langle \eta_\alpha, w\rangle \;=\; \langle v, w\rangle, \qquad \text{for all } v,w\in\mathcal{H}. \tag{9.13}$$

If this is the case, then we have:

$$v \;=\; \sum_{\alpha\in\mathcal{A}} \langle v, \eta_\alpha\rangle \, e_\alpha \;=\; \sum_{\alpha\in\mathcal{A}} \langle v, e_\alpha\rangle \, \eta_\alpha, \qquad \text{for all } v\in\mathcal{H},$$

with convergence in $\mathcal{H}$. Note that, by the polarization identity for sesquilinear forms, we have $K_{e,\eta}(v,w) = \frac{1}{4}\sum_{n=0}^{3} i^n\, K_{e,\eta}(v+i^n w, v+i^n w)$. Therefore, (9.13) holds if and only if it holds for all $v=w\in\mathcal{H}$. Also, it is enough to prove (9.13) for $v=w$ in a dense subspace of $\mathcal{H}$ (cf. [15]).

We have the following general result, which characterizes a class of dual systems for the collections of the form $\{T_{C_p k}\, g_p : p\in\mathcal{P}, k\in\mathbb{Z}^n\}$.

Theorem 9.1. *Let $\{T_{C_p k}\, g_p : p\in\mathcal{P}, k\in\mathbb{Z}^n\}$ and $\{T_{C_p k}\,\gamma_p : p\in\mathcal{P}, k\in\mathbb{Z}^n\}$ be Bessel systems for $L^2(\mathbb{R}^n)$, where $\mathcal{P}$ is countable, $\{g_p\}_{p\in\mathcal{P}}$, $\{\gamma_p\}_{p\in\mathcal{P}}$, are collections of functions in $L^2(\mathbb{R}^n)$ and $\{C_p\}_{p\in\mathcal{P}}\subset GL_n(\mathbb{R})$. Suppose that*

$$\sum_{p\in\mathcal{P}}\sum_{m\in\mathbb{Z}^n}\int_{supp\,\hat f} |\hat f(\xi + C_p^I m)|^2\,\frac{1}{|\det C_p|}\,|\hat g_p(\xi)|^2\,d\xi < \infty. \tag{9.14}$$

and

$$\sum_{p\in\mathcal{P}}\sum_{m\in\mathbb{Z}^n}\int_{supp\,\hat f} |\hat f(\xi + C_p^I m)|^2\,\frac{1}{|\det C_p|}\,|\hat \gamma_p(\xi)|^2\,d\xi < \infty. \tag{9.15}$$

for all $f\in\mathcal{D}$, where $C_p^I = (C_p^t)^{-1}$. Then $\{T_{C_p k}\,\gamma_p : p\in\mathcal{P}, k\in\mathbb{Z}^n\}$ is a dual frame to $\{T_{C_p k}\, g_p : p\in\mathcal{P}, k\in\mathbb{Z}^n\}$ in $L^2(\mathbb{R}^n)$

$$\sum_{p\in\mathcal{P}}\sum_{k\in\mathbb{Z}^n} \langle f, T_{C_p k}\, g_p\rangle\,\langle T_{C_p k}\,\gamma_p, h\rangle \;=\; \langle f, h\rangle \tag{9.16}$$

for all $f,h\in L^2(\mathbb{R}^n)$ if and only if

$$\sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|}\,\overline{\hat g_p(\xi)}\,\hat\gamma_p(\xi+\alpha) = \delta_{\alpha,0} \quad \text{for a.e. } \xi\in\mathbb{R}^n, \tag{9.17}$$

for each $\alpha\in\Lambda$, where δ is the Kronecker delta for $\mathbb{R}^n$, and Λ, $\mathcal{P}_\alpha$ are defined by (2.2) and (2.3).

In order to prove Theorem 9.1, we need the following Lemmas, whose proofs can be easily adapted from those of Lemmas 2.2 and 2.3 (see also [21, Sec.4]). Recall that the dense subspace $\mathcal{D}\subset L^2(\mathbb{R}^n)$ is defined in Section 2.

Lemma 9.2. *Let $C\in GL_n(\mathbb{R})$ and $C^I=(C^t)^{-1}$. If $f\in\mathcal{D}$ and $g,\gamma\in L^2(\mathbb{R}^n)$, then*

$$\sum_{k\in\mathbb{Z}^n} \langle f, T_{Ck}\, g\rangle\,\langle T_{Ck}\,\gamma, f\rangle = \frac{1}{|\det C|}\int_{C^I\mathbb{T}^n} [\hat f,\hat g](\xi; C^I)\,[\hat\gamma,\hat f](\xi;C^I)\,d\xi, \tag{9.18}$$

where $\mathbb{T}^n = [0,1)^n$.

Lemma 9.3. *Let $C \in GL_n(\mathbb{R})$ and $C^I = (C^t)^{-1}$. For each $f \in \mathcal{D}$ and $g, \gamma \in L^2(\mathbb{R}^n)$, the function*

$$K(x) \;=\; \sum_{k \in \mathbb{Z}^n} \langle T_x\, f, T_{Ck}\, g\rangle \, \langle T_{Ck}\, \gamma, T_x\, f\rangle \tag{9.19}$$

is the trigonometric polynomial

$$K(x) = \sum_{m \in \mathbb{Z}^n} \hat{K}(m)\, e^{2\pi i (C^I m)\cdot x},$$

where

$$\hat{K}(m) \;=\; \frac{1}{|\det C|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C^I m)}\, \overline{\hat{g}(\xi)}\, \hat{\gamma}(\xi + C^I m)\, d\xi, \tag{9.20}$$

and only a finite number of these expressions is non-zero.

The following Proposition is the principal result that we shall use to establish Theorem 9.1. The proof is very similar to the proof of Proposition 2.4 and will be omitted. Observe that, unlike Proposition 2.4 where only condition (9.14) was needed, in this case we need both (9.14) and (9.15) in order to show that the generalized Fourier series (9.23) converges absolutely.

Proposition 9.4. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}, \{\gamma_p\}_{p\in\mathcal{P}}$ be collections of functions in $L^2(\mathbb{R}^n)$, $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$, and let $C_p^I = (C_p^t)^{-1}$. Assume that, for $f \in \mathcal{D}$, the conditions (9.14) and (9.15) hold.*

$$\sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\, \hat{f}} |\hat{f}(\xi + C_p^I m)|^2\, \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2\, d\xi < \infty. \tag{9.21}$$

and

$$\sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\, \hat{f}} |\hat{f}(\xi + C_p^I m)|^2\, \frac{1}{|\det C_p|}\, |\hat{\gamma}_p(\xi)|^2\, d\xi < \infty. \tag{9.22}$$

Then, the function

$$w(x) = \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} \langle T_x f, T_{C_p k}\, g_p\rangle \, \langle T_{C_p k}\, \gamma_p, T_x f\rangle$$

is a continuous function that coincides pointwise with its absolutely convergent (almost periodic) Fourier series

$$\sum_{\alpha\in\Lambda} \hat{w}(\alpha)\, e^{2\pi i \alpha\cdot x}\ , \tag{9.23}$$

where

$$\hat{w}(\alpha) \;=\; \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi+\alpha)} \sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|}\, \overline{\hat{g}_p(\xi)}\, \hat{\gamma}_p(\xi+\alpha)\, d\xi, \tag{9.24}$$

and the integral in (9.24) converges absolutely.

Remark. As in Proposition 2.4, the series for $w(x)$ given in Proposition 9.4 is an almost periodic function since these are characterized as uniform limits of generalized trigonometric polynomials (see [1]).

We can now prove Theorem 9.1.

Proof of Theorem 9.1. By the observation at the beginning of this section, it suffices to prove that

$$\sum_{p\in\mathcal{P}}\sum_{k\in\mathbb{Z}^n}\langle f, T_{C_pk}\, g_p\rangle\,\langle T_{C_pk}\,\gamma_p, f\rangle \;=\; \|f\|^2, \tag{9.25}$$

for f in a dense subset of $L^2(\mathbb{R}^n)$. Let us assume that conditions (9.14) and (9.15) hold for all $f\in\mathcal{D}$, where $\mathcal{D}$ is given in Section 2, and that (9.17) is true. By Proposition 9.4,

$$w(x)=\sum_{p\in\mathcal{P}}\sum_{m\in\mathbb{Z}^n}\langle T_x\, f, T_{C_pm}\, g_p\rangle\,\langle T_{C_pm}\,\gamma_p, T_x\, f\rangle=\sum_{\alpha\in\Lambda}\hat{w}(\alpha)\, e^{2\pi i\alpha\cdot x},$$

where the last series converges absolutely (thus, $w(x)$ is continuous) and, by (9.17),

$$\hat{w}(\alpha)\;=\;\Big(\int_{\mathbb{R}^n}\hat{f}(\xi)\,\overline{\hat{f}(\xi+\alpha)}\,d\xi\Big)\,\delta_{\alpha,0}$$

for each $f\in\mathcal{D}$. Then equation (9.25) follows by letting $x=0$.

Now let us assume that equation (9.25) holds for all $f\in L^2(\mathbb{R}^n)$. By Proposition 9.4, if $f\in\mathcal{D}$, then the function $z(x)=w(x)-\|f\|^2$ is continuous and equals an absolutely convergent (generalized) trigonometric series whose coefficients are

$$\hat{z}(0)=\hat{w}(0)-\|f\|^2,\qquad\text{and}\qquad \hat{z}(\alpha)=\hat{w}(\alpha),\ \alpha\neq 0.$$

Since $z(x)=0$, it follows from Lemma 2.5 that all coefficients $\hat{z}(\alpha)$ must be 0. Thus for $\alpha\in\Lambda$ and $f\in\mathcal{D}$

$$\int_{\mathbb{R}^n}\hat{f}(\xi)\,\overline{\hat{f}(\xi+\alpha)}\Big(\sum_{p\in\mathcal{P}_\alpha}\frac{1}{|\det C_p|}\,\overline{\hat{g}_p(\xi)}\,\hat{\gamma}_p(\xi+\alpha)\Big)\,d\xi=\delta_{\alpha,0}\,\|f\|^2, \tag{9.26}$$

Consider the case $\alpha=0$ and let

$$s_0(\xi)=\sum_{p\in\mathcal{P}}\frac{1}{|\det C_p|}\,\overline{\hat{g}_p(\xi)}\,\hat{\gamma}_p(\xi).$$

By (9.14) and (9.15), s_0 is locally integrable. Choose ξ_0 to be a point of differentiability of the integral of this function. Letting $B(\epsilon)$ denote the ball of radius $\epsilon>0$ about the origin, define f_ϵ by

$$\hat{f}_\epsilon(\xi)=\frac{1}{\sqrt{|B(\epsilon)|}}\,\chi_{B(\epsilon)}(\xi-\xi_0)\,.$$

Then $\|f_\epsilon\|_2=1$ and $f_\epsilon\in\mathcal{D}$. By (9.26) with $f=f_\epsilon$ we have

$$1=\lim_{\epsilon\to 0}\int_{|\xi-\xi_0|\le\epsilon}\frac{1}{|B(\epsilon)|}\,s_0(\xi)\,d\xi=s_0(\xi_0).$$

This shows that $s_0(\xi) = 1$, a.e. $\xi \in \mathbb{R}^n$, and (9.17) is satisfied for $\alpha = 0$.
When $\alpha \neq 0$, let

$$s_\alpha(\xi) = \sum_{p \in \mathcal{P}(\alpha)} \frac{1}{|\det C_p|} \overline{\hat{g}_p(\xi)}\, \hat{\gamma}_p(\xi + \alpha).$$

By the polarization of (9.26), we have

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{h}(\xi + \alpha)}\, s_\alpha(\xi)\, d\xi = 0 \tag{9.27}$$

for all $f, h \in \mathcal{D}$. By Schwarz's inequality and conditions (9.14), (9.15), we have that s_α is locally integrable. We can choose, again, a point of differentiability ξ_0 of the integral of s_α, and choose f_ϵ and h_ϵ such that

$$\hat{f}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}} \chi_{B(\epsilon)}(\xi - \xi_0), \qquad \hat{h}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}} \chi_{B(\epsilon)}(\xi - \xi_0 - \alpha).$$

Hence $\|f_\epsilon\|_2 = \|g_\epsilon\|_2 = 1$, $f_\epsilon\ , g_\epsilon \in \mathcal{D}$ and by (9.27),

$$0 = \lim_{\epsilon \to 0} \int_{|\xi - \xi_0| \leq \epsilon} \frac{1}{|B(\epsilon)|}\, s_\alpha(\xi)\, d\xi = s_\alpha(\xi_0).$$

Hence $s_\alpha(\xi) = 0$, a.e. $\xi \in \mathbb{R}^n$, and (9.17) is satisfied for $\alpha \neq 0$. □

The application of Theorem 9.1 to the Gabor systems $\mathcal{G}_{B,C}(G)$, defined by (3.2), yields the following characterization of Gabor dual frames, known as the Wexler-Raz theorem (cf. [20, 30, 21]). Our proof, which is adapted from [21], will only be sketched.

Theorem 9.5 (Wexler-Raz). *Let $G = \{g^1, \cdots, g^L\}$, $\Gamma = \{\gamma^1, \cdots, \gamma^L\} \subset L^2(\mathbb{R}^n)$, $B, C \in GL_n(\mathbb{R})$, and assume that $\mathcal{G}_{B,C}(G)$ and $\mathcal{G}_{B,C}(\Gamma)$ are Bessel systems for $L^2(\mathbb{R}^n)$. Then the system $\mathcal{G}_{B,C}(\Gamma)$ is a dual system to $\mathcal{G}_{B,C}(G)$*

$$\sum_{\ell=1}^{L} \sum_{k,m \in \mathbb{Z}^n} \langle f, T_{Ck}\, M_{Bm}\, g^\ell \rangle \langle T_{Ck}\, M_{Bm}\, \gamma^\ell, h \rangle \;=\; \langle f, h \rangle \qquad \text{for all } f, h \in L^2(\mathbb{R}^n) \tag{9.28}$$

if and only if

$$\sum_{\ell=1}^{L} \langle g^\ell, T_{B^I v}\, M_{C^I u}\, \gamma^\ell \rangle = |\det B|\, |\det C|\, \delta_{u,0}\, \delta_{v,0} \tag{9.29}$$

for each $u,\ v \in \mathbb{Z}^n$, where δ is the product Kronecker delta in $\mathbb{Z}^n$, $B^I = (B^t)^{-1}$ and $C^I = (C^t)^{-1}$.

Proof. We apply Theorem 9.1 with $g_p = M_{Bp}\, g$, $\gamma_p = M_{Bp}\, \gamma$, $p \in \mathcal{P} = \mathbb{Z}^n$ and $C_p = C$. An argument similar to the proof of Theorem 3.2 shows that, if $\mathcal{G}_{B,C}(G)$ and $\mathcal{G}_{B,C}(\Gamma)$ are Bessel systems for $L^2(\mathbb{R}^n)$, then

$$\sum_{\ell=1}^{L} \sum_{k,m \in \mathbb{Z}^n} \langle f, T_{Ck}\, M_{Bm}\, g^\ell \rangle \langle T_{Ck}\, M_{Bm}\, \gamma^\ell, h \rangle \;=\; \langle f, h \rangle \qquad \text{for all } f, h \in L^2(\mathbb{R}^n)$$

if and only if

$$F(\xi) = \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} \frac{1}{|\det C|}\, \hat{g}^{\ell}(\xi - Bk)\, \overline{\hat{\gamma}^{\ell}(\xi - Bk + C^I m)} = \delta_{m,0}$$

for a.e. $\xi \in \mathbb{R}^n$, all $m \in \mathbb{Z}^n$.

The proof then follows by expanding the $B\mathbb{Z}^n$- periodic function $F(\xi)$ into a Fourier series, as in the argument used in [21, theorem 6.1]. $\square$.

The application of Theorem 9.1 to the affine systems $\mathcal{F}_A(\Psi)$, defined by (5.2), yields the following characterization of affine dual systems, whose proof is similar to the proof of Theorem 2.1. This theorem generalizes previous results about affine dual systems, such as those in [15, 2, 7].

Theorem 9.6. *Let $\Psi = \{\psi^1, \cdots, \psi^L\}, \Phi = \{\phi^1, \cdots, \phi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ such that the matrix $B = A^t$ is expanding for a subspace F of $\mathbb{R}^n$. Assume that the systems $\mathcal{F}_A(\Psi)$ and $\mathcal{F}_A(\Phi)$ are Bessel systems for $L^2(\mathbb{R}^n)$. Then the system $\mathcal{F}_A(\Phi)$ is a dual system to $\mathcal{F}_A(\Psi)$*

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \langle f, D_A^j\, T_k\, \psi^\ell\rangle \langle D_A^j\, T_k\, \phi^\ell, h\rangle \;=\; \langle f, h\rangle \qquad \text{for all } f, h \in L^2(\mathbb{R}^n) \tag{9.30}$$

if and only if

$$\sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_\alpha} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\phi}^\ell(B^{-j}(\xi+\alpha))} = \delta_{\alpha,0} \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{9.31}$$

and all $\alpha \in \Lambda = \bigcup_{j\in\mathbb{Z}} B^j(\mathbb{Z}^n)$, where, for $\alpha \in \Lambda$, $\mathcal{P}_\alpha = \{j \in \mathbb{Z} :\, B^{-j}\alpha \in \mathbb{Z}^n\}$.

Proof. Recall that $D_A^j\, T_k\, \psi^\ell = T_{A^{-j}k}\, D_A^j\, \psi^\ell$. We are going to apply Theorem 9.1 with

$$\mathcal{P} = \big\{(j,\ell) : j \in \mathbb{Z},\, \ell = 1, 2, \ldots, L\big\},$$

$$g_p \equiv g_{(j,l)} = D_A^j\, \psi^\ell,\ \gamma_p \equiv \gamma_{(j,l)} = D_A^j\, \phi^\ell, \quad \text{and} \quad C_p \equiv C_{(j,\ell)} = A^{-j} \text{ for all } \ell = 1, \ldots, L.$$

Since we have that $\hat{g}_p(\xi) = (D_A^j\psi^\ell)^\wedge(\xi) = |\det B|^{-j/2}\hat{\psi}^\ell(B^{-j}\,\xi)$, and $\hat{\gamma}_p(\xi) = (D_A^j\phi^\ell)^\wedge(\xi) =$ $= |\det B|^{-j/2}\hat{\phi}^\ell(B^{-j}\,\xi)$, then (9.31) follows from (9.17) in Theorem 9.1, provided the conditions (9.14) and (9.15) in this Theorem are satisfied. Therefore, all that it is left to prove is that (9.14) and (9.15) are satisfied in this particular case. Thus, we need to show that:

$$L(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\mathrm{supp}\hat{f}} |\hat{f}(\xi + B^j m)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi < \infty \tag{9.32}$$

and

$$J(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\mathrm{supp}\,\hat{f}} |\hat{f}(\xi + B^j m)|^2\, |\hat{\phi}^\ell(B^{-j}\xi)|^2\, d\xi < \infty, \tag{9.33}$$

for f in an appropriate dense set of $L^2(\mathbb{R}^n)$. Like in the proof of Theorem 2.1, the dense set we choose is

$$\mathcal{D}_E = \{f \in \mathcal{D} : (\mathrm{supp}\hat{f}) \cap E = \emptyset\}$$

where $\mathcal{D} = \{f \in L^2(\mathbb{R}^n) \colon \hat{f} \in L^\infty(\mathbb{R}^n)$ and $\mathrm{supp}\,\hat{f}$ is compact$\}$, and E is a complementary subset to F as in Definition 5.1. The proof that $L(f) < \infty$ and $J(f) < \infty$ now follows from Proposition 5.6. $\square$

References

[1] Bochner, S. Beiträge zur Theorie der fastperiodischen Funktionen. II, *Math. Ann.*, **96**, 383–409, (1927).

[Boh32] Bohr, H., Fastperiodische Funktionen, Springer Verlag, Berlin, (1932).

[Cor68] Corduneanu, C. Almost-Periodic Functions, Interscience Publishers, New York, (1968).

[2] Bownik, M. A Characterizations of Affine Dual Frames in $L^2(\mathbb{R}^n)$, *Appl. Comput. Harmon. Anal.*, **8**(2), 203-221, (2000).

[3] Bownik, M. On Characterizations of Multiwavelets in $L^2(\mathbb{R}^n)$, *Proc. Am. Math. Soc.*, **129**, 3265-3274, (2001).

[4] Bownik, M. Quasi affine systems and the Calderón condition, preprint, (www.math.lsa.umich.edu/∼marbow/papers.html), (2001).

[5] Bownik, M. and Speegle, D. The wavelet dimension function for real dilations and dilations admitting non-MSF wavelets, preprint, (www.math.lsa.umich.edu/∼marbow/papers.html), (2001).

[6] Calogero, A. A Characterization of Wavelets on General Lattices, *J. Geom. Anal.*, **10**(4), 597–622, (2000).

[7] Chui, C., Czaja, W., Maggioni, M., Weiss, G. Characterization of General Tight Wavelets Frames with Matrix Dilations and Tightness Preserving Oversampling, *J. Fourier Anal. Appl.*, to appear, (2001).

[8] Chui, C., and Shi, X. Orthonormal Wavelets and Tight Frames with Arbitrary Real Dilations, *Appl. Comp. Harmonic Anal.*, **9**, 243-264, (2000).

[9] Chui, C., Shi, X., and Stöckler, J. Affine Frames, Quasi-Affine Frames and Their Duals, *Adv. Comp. Math.*, **8**, 1-17, (1998).

[10] Czaja, W. Characterizations of Gabor Systems via the Fourier Transform, *Collect. Math.*, **51** 2, 205-224, (2000).

[11] DeBoor C., DeVore, R.A., Ron A. The Structure of Finitely Generated Shift-Invariant Spaces in $L^2(\mathbb{R}^n)$, *J. Funct. Anal.*, **119**, 37–78, (1994).

[12] Daubechies, I., Landau, H., Landau, Z. Gabor Time–Frequency Lattices and the Wexler–Raz Identity, *J. Fourier Anal. Appl.*, **1**, 437–478, (1995).

[13] Dai, X., Larson, D.R. Wandering vectors for Unitary Systems of Orthogonal Wavelets, *Memoirs of the A.M.S*, **640**, (1998).

[14] Dai, X., Larson, D.R., Speegle, D.M. Wavelet sets in $\mathbb{R}^n$ II, *Wavelets, multiwavelets and their applications (San Diego, CA, 1997) Contemp. Math.,(Amer. Math. Soc)*, **216**, 15-40, (1998).

[15] Frazier, M., Garrigós, G., Wang, K., and Weiss, G. A characterization of functions that generate wavelets and related expansions, *J. Fourier Anal. and Appl.*, **3**, 883-906, (1997).

[16] Gripenberg, G. A necessary and sufficient condition for the existence of a father wavelet, *Studia Math.*, **114**, 297-226, (1995).

[17] Han, B. On dual wavelet tight frames, *Appl. Comput. Harm. Anal*, **4**, 380-413, (1997).

[18] Hernández, E. and Weiss, G. *A First Course on Wavelets*, CRC Press, Boca Raton FL, 1996.

[19] Janssen, A.J.E.M. Signal analytic proof of two basic results on lattice expansions, *Appl. Comp. Harm. Anal.*, **1(4)**, 350-354, (1994).

[20] Janssen, A.J.E.M. Duality and Biorthogonality for Weyl-Heisenberg Frames, *J. Fourier Anal. Appl.*, **1(4)**, 403-436, (1995).

[21] Labate, D. A Unified Characterization of Reproducing Systems Generated by a Finite Family, *J. Geom. Anal.*, to appear, (www.math.wustl.edu/∼dlabate/publications.html), (2001).

[22] Landau, H. On the density of phase-space expansions, *IEEE Trans. Info. Theory*, **39**, 1152-1156, (1993).

[23] Laugesen, R.S. Completeness of orthonormal wavelet systems for arbitrary real dilations, *Appl. Comput. Harmonic Anal.*, **11**, 455-473, (2001).

[24] Laugesen, R.S. Translational averaging for completeness, characterization and oversampling of wavelets, *Collec. Math.*, to appear, (www.math.uiuc.edu/∼laugesen/publications.html), (2002).

[25] Laugesen, R.S., Weaver, N., Weiss, G.L. and Wilson, E.N. A characterization of the higher dimensional groups associated with continuous wavelets, *J. Geom. Anal.*, to appear, (2001).

[26] Lemarié-Rieusset, P.G. Projecteurs invariants, matrices de dilatation, ondelettes et analyses multi-résolutions, *Rev. Mat. Iberoamericana*, **10**, 283-347, (1994)

[27] Rieffel, M.A. Von Neuman algebras associated with pair of lattices in Lie groups, *Math. Ann.*, **257**, 403-418, (1981).

[28] Ron, A., and Shen, Z. Frames and Stable Bases for Shift-Invariant Subspaces of $L_2(\mathbb{R}^d)$, *Can. J. Math.*, **47**, 1051-1094, (1995).

[29] Ron, A., and Shen, Z. Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator, *J. Funct. Anal.*, **148**, 408-447, (1997).

[30] Ron, A., and Shen, Z. Weyl-Heisenberg frames and Riesz bases in $L^2(\mathbb{R}^d)$, *Duke Math. J.*, **89**, 237-282, (1997).

[31] Rudin, W. *Functional Analysis*, McGraw Hill Book Company, New York NY, 1973.

[32] Rzeszotnik, Z. *Characterization theorems in the theory of wavelets*, Ph. D. Thesis, Washington University in St. Louis, 2000.

[33] Rzeszotnik, Z. Calderón's condition and wavelets, *Collec. Math.*, **52**, 181-191, (2001)

[34] Soardi, P. and Weiland, D. Single wavelets in n-dimensions, *J. Fourier Anal. Appl.*, **4**, 205-212, (1998)

[35] Wang, X. *The study of wavelets from the properties of their Fourier transforms*, Ph.D. Thesis, Washington University in St. Louis, 1995.

[36] Zaidman, S. *Almost-Periodic Functions in Abstract Spaces*, Pitman Advanced Publ., Boston, 1985.

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Tight Frame Wavelets, their Dimension Functions, MRA Tight Frame Wavelets and Connectivity Properties

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Abstract

This is a continuation of our study of *Generalized Low Pass Filters and MRA Frame Wavelets*. In this first study we concentrated on the construction of such functions. Here we are particularly interested in the role played by the Dimension Function. In particular we characterize all semi-orthogonal Tight Frame Wavelets (TFW) by showing that they correspond precisely to those for which the dimension function is non-negative integer-valued. We also show that a TFW arises from our MRA construction if and only if the dimension of a particular linear space is either zero or one. We present many examples. In addition we obtain a result concerning the connectivity of TFW's that are MSF tight frame wavelets.

1 Introduction

The "classical" MRA wavelets are probably the most important class of orthonormal wavelets. Many of the better known examples as well as those

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often used in applications belong to this class. Thus, it was a natural question to find necessary and sufficient conditions for an orthonormal wavelet to be an MRA wavelet. An interesting approach to this involves the dimension function. In fact we have (see Chapter 7 of [HW])

Theorem 1.1 *An orthonormal wavelet is an MRA wavelet if and only if its dimension function equals* 1 *almost everywhere.*

From an application point of view there are at least two fundamental features that MRA wavelets possess: (1) the reconstruction property (shared with all orthonormal wavelets) that enables us to "recover" (or "synthesize") an "analysed" function, and (2) the low pass filter which allows us to apply particularly convenient algorithms for the analysis of a function. MRA wavelets, however, form orthonormal basis, which is often a desirable feature, but it does impose some restrictions. For example, there are useful "filters", such as $m(\xi) = \frac{1}{2}(1 + e^{3i\xi})$, that do not produce orthonormal basis, nevertheless, they do produce systems that have the reconstruction property, as well as many other useful features. It is natural, therefore, to develop a theory involving more general filters that do, indeed, produce systems having these properties. A natural setting for such a theory is provided by *frames* (see Ch. 8 of [HW]). Several authors have considered this problem and have shown how more general filters produce such frames. A successful development of these ideas is provided by the papers [BL] and [BT]. These results, however, do involve certain restrictions and technical assumptions such as semi-orthogonality (see Section 2 for a definition of this notion). In particular, they exclude the use of the filter we described above. Very recently, we successfully developed a somewhat more general approach to this problem (see [PSWX]). Building up from a careful analysis of filters in [PSW] we developed a theory of *Tight Frame Wavelets* (TFW) that also includes the MRA concept and obtained a general theory of *MRA Tight Frame Wavelets* (MRA TFW). We refer the reader to [PSWX] for the technical details, though we shall explain some of the basic notions in this paper as we use them. Having Theorem 1.1 in mind, certain questions are simply forced upon us: What are necessary and sufficient conditions for a TFW to be an MRA TFW? Does the *dimension function*

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 \tag{1.2}$$

for a TFW enjoy some of the properties it has when ψ is an orthonormal wavelet? In particular, can it be used to characterize the MRA TFW's (as is the case for orthonormal wavelets, as is stated in Theorem 1.1)? It is clear that D_ψ is a 2π-periodic function in $L^1([-\pi, \pi))$. What role does semi-orthogonality play in this framework? What values does D_ψ assume? An interesting picture will emerge that shows that Theorem 1.1 is just a special case of a much wider class of results. The detailed description of this picture, which will contain, in particular, the answers to the above questions, is the central theme of this article. A list of the main results will be presented in Section 3. In the course of these developments we shall obtain some interesting

auxilliary results involving the question of when does a function ψ generate a frame for the subspace which is the span of its integral translates. This involves a re-examination of some of the results in [PSWX], Section 5, and ties in with some recently obtained results for "shift invariant" spaces in [WW]. Since these results also serve as a technical prerequisite for Section 3, they are collected in Section 2. In Section 4, we turn our attention to the connectivity of the set of MRA TFW's. It was shown in [WUTAM] that the set of MRA wavelets is connected. This result was obtained from the properties of "multipliers". In [PSWX], we presented a complete description of the TFW multipliers. It is natural to expect that from these one obtains the analogous results for MRA TFW's. Certain difficulties, however, arise in this more general case (we encountered some in Section 4 of [PSWX]). There is also a result of D. Speegle [S] that shows that the MSF wavelets are connected. We shall examine this situation in the TFW setting.

2 The periodization function and W_0 frames

The main notion studied in this paper is the dimension function (see (1.2)) of a TFW ψ. This name is not as appropriate, at this level of generality, as is the case when studying orthonormal wavelets, when its value, in fact, is the dimension of a linear space. Nevertheless, we shall continue using this name even though $D_\psi(\xi)$ need not be such a dimension and, in fact, may not even be integer valued. Closely related to the dimension function is the *periodization function*, σ_ψ, of a function $\psi \in L^2(\mathbb{R})$. It is defined by

$$\sigma_\psi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 \qquad (2.1)$$

for $\xi \in \mathbb{R}$. As in the case for D_ψ, σ_ψ is a 2π-periodic function in $L^1([-\pi,\pi))$. We should declare that the Fourier transform we are using is given by the equality

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$$

for $f \in L^1(\mathbb{R})$. Let us list two properties of σ_ψ and D_ψ that follow easily from their definitions:

$$\int_{-\pi}^{\pi} D_\psi(\xi) d\xi = ||\hat{\psi}||_2^2 = \int_{-\pi}^{\pi} \sigma_\psi(\xi) d\xi, \tag{2.2}$$

$$D_\psi(\xi) + D_\psi(\xi + \pi) = D_\psi(2\xi) + \sigma_\psi(2\xi). \tag{2.3}$$

The results in this paper are stated in terms of frames. Hence, we remind the reader that a frame in a Hilbert space $(H, < \cdot, \cdot >)$ is a family $\{\varphi_n\}, n \in \Delta$, of elements of H for which there exist two positive constants A and B such that

$$A||f||^2 \leq \sum_{n \in \Delta} | < f, \varphi_n > |^2 \leq B||f||^2 \tag{2.4}$$

for all $f \in H$. Though one can define this concept for more general indexing sets, we only consider the case where Δ is countable. In fact, we shall restrict ourselves mostly to the case $A = B$; in which case, the family $\{\varphi_n\}, n \in \Delta$, is called a *tight frame*. In this case we can always normalize the members of the collection so that $A = 1 = B$. As was done in [PSWX] we will often tacitly assume this to be the case. For the general properties of frames we refer the reader to [HW], Chapter 8. Some more technical properties can be found in [PSWX] and the references cited there. We are interested in frames generated by a single function by translations and dilations. As is the case in [PSWX] a function $\psi \in L^2(\mathbb{R})$ is a *tight frame wavelet (TFW)* if and only if the system $\{\psi_{jk}(x)\} \equiv \{2^{\frac{j}{2}}\psi(2^j x - k)\}, j, k \in \mathbb{Z}$, is a tight frame (with $A = 1 = B$) for $H = L^2(\mathbb{R})$. By definition this means that

$$||f||_2^2 = \sum_{j,k\in\mathbb{Z}} | < f, \psi_{jk} > |^2 \tag{2.5}$$

for all $f \in L^2(\mathbb{R})$. It is not hard to see that this is equivalent to

$$f = \sum_{j,k\in\mathbb{Z}} < f, \psi_{jk} > \psi_{j,k} \tag{2.6}$$

for all $f \in L^2(\mathbb{R})$, where the sum converges unconditionally in $L^2(\mathbb{R})$. A deeper result is

Proposition 2.7 *$\psi \in L^2(\mathbb{R})$ is a TFW if and only if*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1 \qquad a.e. \tag{2.8}$$

and

$$t_q(\xi) = \sum_{j\geq 0} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi + 2q\pi))} = 0 \qquad a.e. \tag{2.9}$$

whenever q is an odd integer.

See Chapter 7 of [HW] for an account of all these results. The main purpose of this article is to study the dimension function of a TFW. In particular, we shall be interested in those properties of D_ψ that determine whether ψ is an MRA TFW. In order to present a more complete account of what this situation means, we need to define a few concepts. A TFW is said to be *semi-orthogonal* if and only if $\psi_{j_1k_1}$ is orthogonal to $\psi_{j_2k_2}$ whenever $j_1 \neq j_2$ (k_1 and k_2 are any two integers). This is equivalent to the orthogonality of the subspaces W_{j_1} and W_{j_2} if $j_1 \neq j_2$, where

$$W_j = \overline{span\{\{\psi_{jk}\} : k \in \mathbb{Z}\}}. \tag{2.10}$$

as j ranges throughout $\mathbb{Z}$. Let us now recall what is an MRA TFW, a concept we defined in [PSWX]. As was done in [PSW], we denote by $\tilde{\mathbf{F}}$ the set of *generalized filters*; that is, $m \in \tilde{\mathbf{F}}$ if it is a 2π-periodic measurable function satisfying

$$|m(\xi)|^2 + |m(\xi + \pi)|^2 = 1 \tag{2.11}$$

for a. e. $\xi \in \mathbb{R}$. A function $\varphi \in L^2(\mathbb{R})$ is called a *pseudo-scaling function* if and only if there exists $m \in \tilde{\mathbf{F}}$ (not necessarily unique) such that

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi) \tag{2.12}$$

for a. e. $\xi \in \mathbb{R}$. A TFW ψ is an MRA TFW if and only if there exists a pseudo-scaling function φ and a corresponding generalized filter m such that

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\overline{m(\frac{\xi}{2} + \pi)}\hat{\varphi}(\frac{\xi}{2}) \tag{2.13}$$

for a. e. $\xi \in \mathbb{R}$. It turns out that if ψ is an MRA TFW, then m has to be more than just a generalized filter; m has to be a *generalized low pass filter*, which means that the Lebesgue measure of the set

$$N_0(|m|) = \{\xi \in \mathbb{R} : \lim_{n\to\infty} \hat{\varphi}_{|m|}(2^{-n}\xi) = 0\}$$

is 0, where

$$\hat{\varphi}_{|m|}(\xi) = \prod_{j=1}^{\infty} |m(2^{-j}\xi)|. \tag{2.14}$$

A few comments are in order. The partial products in (2.14) are non-increasing and do not exceed 1; thus, they do converge to a non-negative function $\hat{\varphi}_{|m|}$ whose values are less than or equal to 1 a. e. In the definition of $N_0(|m|)$ we could have restricted ξ to belong to $I = [-\pi, \pi) \setminus [-\frac{\pi}{2}, \frac{\pi}{2})$; as we shall see, this is a natural restriction. Furthermore, for such φ, ψ and m we have

$$|\hat{\varphi}(\xi)| = \hat{\varphi}_{|m|}(\xi) \tag{2.15}$$

and

$$|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)|^2 \tag{2.16}$$

for a. e. $\xi \in \mathbb{R}$ (see Section 2 of [PSWX] and [PSW] for proofs and details). Recall also that the MRA TFW's are precisely those $\psi \in L^2(\mathbb{R})$ that can be constructed from a generalized low pass filter as described in Section 2 of [PSWX] (see, in particular, Theorems 2.18 and 2.19 of that paper). What are the properties of σ_ψ when ψ is a TFW? We shall see that the answers to this question are important for determining the properties of D_ψ. The space W_0 is generated by the integral translates of ψ (see (2.10) when $j = 0$). For a general $\psi \in L^2(\mathbb{R})$ (not necessarily a TFW), let us consider $W(\psi) \equiv \overline{span\{\psi(\cdot - k) : k \in \mathbb{Z}\}}$. This space is sometimes called the *principal shift-invariant space* generated by ψ. It is clear that if $\mathcal{A}(\psi)$ is the algebraic span of $\{\psi(\cdot - k), k \in \mathbb{Z}\}$, then, $f \in \mathcal{A}(\psi)$ if and only if $\hat{f} = t\hat{\psi}$, where t is a trigonometric polynomial, $t(\xi) = \sum_{finite} a_k e^{ik\xi}$. Moreover,

$$||f||_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |t(\xi)|^2 |\hat{\psi}(\xi)|^2 d\xi = \int_{-\pi}^{\pi} |t(\xi)|^2 \sigma_\psi(\xi) \frac{d\xi}{2\pi}. \tag{2.17}$$

It follows that

Proposition 2.18 *$f \in W(\psi)$ if and only if*

$$\hat{f}(\xi) = t(\xi)\hat{\psi}(\xi) \tag{2.19}$$

where $t \in L^2(\mathbb{T}, \sigma_\psi(\xi)\frac{d\xi}{2\pi})$ and

$$||f||_2^2 = ||t||^2_{L^2(\mathbb{T}, \sigma_\psi(\xi)\frac{d\xi}{2\pi})}. \tag{2.20}$$

Remark: This proposition follows by completing $\mathcal{A}(\psi)$ and its isometric image consisting of all trigonometric polynomials t endowed with the norm defined by the last expression in (2.17), where $\mathbb{T}$ denotes the torus, identified with the interval $[-\pi, \pi]$ together with addition modulo 2π. It is useful to think of the elements $t \in L^2(\mathbb{T}, \sigma_\psi(\xi)\frac{d\xi}{2\pi})$ to be 0 outside $U = U_\psi = \{\xi \in \mathbb{T}, \sigma_\psi(\xi) > 0\}$ and extended to $\mathbb{R}$ so that they are 2π-periodic. When ψ is a TFW, then the space W_0 we defined in (2.10) is the space $W(\psi)$. We will be interested in examining what type of spanning set for W_0 is $\{\psi(\cdot - k)\} = \{\psi_{0k}\}, k \in \mathbb{Z}$. If ψ is an orthonormal wavelet, then $\{\psi(\cdot - k)\}$ is an orthonormal basis for W_0. What happens if ψ is a TFW? The following facts will be useful for studying this question. It is not hard to see that we can find a $\varphi \in W(\psi)$ such that $\{\varphi(\cdot - k)\}, k \in \mathbb{Z}$, is a tight frame (with constant 1) for $W(\psi)$. Indeed, let $s(\xi) = (\sigma_\psi(\xi))^{-\frac{1}{2}}$ if $\xi \in U_\psi$ and $s(\xi) = 0$ if $\xi \notin U_\psi$. Then, clearly, $s \in L^2(\mathbb{T}, \sigma_\psi(\xi)\frac{d\xi}{2\pi})$. Straight forward calculations show that φ defined by $\hat{\varphi}(\xi) = s(\xi)\hat{\psi}(\xi)$ provides us with the desired function. Moreover,

$$\sigma_\varphi(\xi) = \chi_{U_\psi}(\xi) \tag{2.21}$$

and $U_\psi = U_\varphi$. It is also clear that the mapping $\mathcal{T} : L^2(\mathbb{T}, \chi_U(\xi)\frac{d\xi}{2\pi}) \to L^2(\mathbb{T}, \sigma_\psi(\xi)\frac{d\xi}{2\pi})$ where $\mathcal{T}t = st$ for $t \in L^2(\mathbb{T}, \chi_U(\xi)\frac{d\xi}{2\pi})$, is an isometry onto. Equality (2.21) characterizes those $\varphi \in L^2(\mathbb{R})$ such that $\{\varphi(\cdot - k)\}, k \in \mathbb{Z}$, is a tight frame (of constant 1) for $W(\varphi)$. This result can be extended to the following known theorem; however, this proof will be interesting for us later.

Theorem 2.22 *$\{\psi(\cdot - k)\}, k \in \mathbb{Z}$ is a frame with constants A, B for $W(\psi)$ if and only if*

$$A\chi_U \leq \sigma_\psi \leq B\chi_U \qquad a.e. \tag{2.23}$$

Proof: Observe that if $f \in W(\psi)$ then

$$< f, \psi(\cdot - k) >= \int_{\mathbb{T}} t(\xi)\sigma_\psi(\xi)e^{ik\xi}\frac{d\xi}{2\pi} \tag{2.24}$$

and, thus, $t(\xi)\sigma_\psi(\xi) = \sum_{k\in\mathbb{Z}} < f, \psi(\cdot - k) > e^{-ik\xi}$. Consequently,

$$\sum_{k\in\mathbb{Z}} | < f, \psi(\cdot - k) > |^2 = \int_{\mathbb{T}\cap U} |t(\xi)|^2(\sigma_\psi(\xi))^2\frac{d\xi}{2\pi}. \tag{2.25}$$

Now, let us assume (2.23) is true; using (2.17) and (2.25), we then have

$$\begin{aligned} A||f||_2^2 &= \frac{A}{2\pi}\int_{\mathbb{T}\cap U}|t(\xi)|^2\sigma_\psi(\xi)d\xi \leq \frac{1}{2\pi}\int_{\mathbb{T}\cap U}|t(\xi)|^2(\sigma_\psi(\xi))^2 d\xi \\ &\leq \frac{B}{2\pi}\int_{\mathbb{T}\cap U}|t(\xi)|^2\sigma_\psi(\xi)d\xi = B||f||_2^2. \end{aligned}$$

Then, by (2.25), we see that our system is a frame for $W(\psi)$. Conversely, suppose the system in question is a frame for $W(\psi)$, then, using (2.17) and (2.25) again, we have

$$A\int_{\mathbb{T}\cap U}|t(\xi)|^2\sigma_\psi(\xi)d\xi \leq \int_{\mathbb{T}\cap U}|t(\xi)|^2(\sigma_\psi(\xi))^2 d\xi \leq B\int_{\mathbb{T}\cap U}|t(\xi)|^2\sigma_\psi(\xi)d\xi$$

for all $t \in L^2(\mathbb{T}, \sigma_\psi(\xi)\frac{d\xi}{2\pi})$. First choose t such that $|t(\xi)|^2 = \chi_E(\xi)$, where $E = \{\xi \in \mathbb{T}\cap U : \sigma_\psi(\xi) < A\}$, and, secondly, t such that $|t(\xi)|^2 = \chi_F(\xi)$, where $F = \{\xi \in \mathbb{T}\cap U : \sigma_\psi(\xi) > B\}$ and we easily deduce that both E and F have measure 0. ■

A consequence of the argument we used to prove this result is that if $\{\psi(\cdot - k)\}, k \in \mathbb{Z}$, is a *Bessel system* (with constant 1), which means that

$$\sum_{k\in\mathbb{Z}} |< f, \psi(\cdot - k) >|^2 \leq ||f||_2^2 \quad \text{for all } f \in W(\psi), \tag{2.26}$$

then $\sigma_\psi(\xi) \leq 1$. In particular, if ψ is a TFW this last inequality is then true.

Let us return to the study of the space W_0 where ψ is a TFW. The question concerning whether ψ generates a frame for $W_0 = W(\psi)$ does not have a "simple" answer. Let us consider some examples.

Example 2.27 Let $\psi(x) = \chi_{[-2,-\frac{1}{2})}(x) - \chi_{[-\frac{1}{2},1)}(x)$. Then ψ is an MRA TFW (as was observed in (5.38) of [PSWX]) and satisfies equality (2.13) with $m(\xi) = \frac{1}{2}(1 + e^{3i\xi})$ and $\hat{\varphi}(\xi) = \frac{i}{3\xi}(1 - e^{3i\xi})$; thus,

$$\hat{\psi}(\xi) = \frac{e^{i\frac{\xi}{2}}(1 - e^{-\frac{3}{2}i\xi})(1 - e^{\frac{3}{2}i\xi})}{-3i\xi}.$$

The dimension function in this case is

$$D_\psi(\xi) = \frac{1}{9}(1 + 2\cos\xi)^2. \tag{2.28}$$

This equality follows from

$$\sum_{k\in\mathbb{Z}} \frac{1}{(\xi + 2k\pi)^2} = \frac{1}{4\sin^2\frac{\xi}{2}}$$

(see page 144 of [HW]) and simple trigonometric identities. Also,

$$\sigma_\psi(\xi) = \frac{1}{9}(1 + 4\sin^2\xi) \tag{2.29}$$

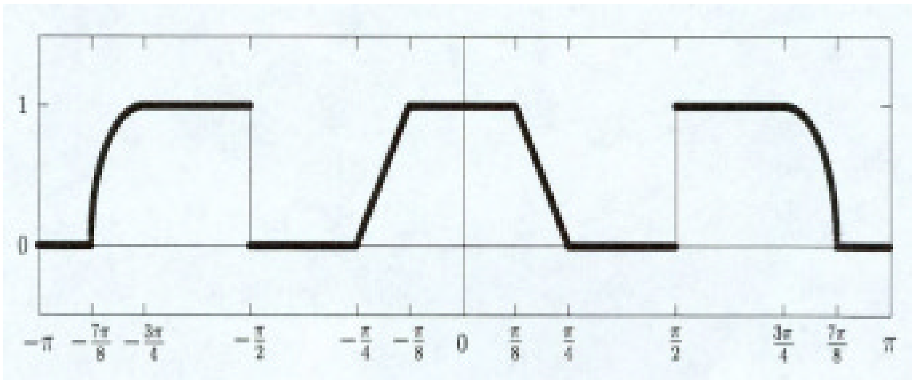
which follows from (2.3) and (2.28). It is clear that $U = U(\psi) = \{\xi \in \mathbb{R} : \sigma_\psi(\xi) > 0\} = \mathbb{R}$ a. e. (we sometimes restrict this periodic set to $\mathbb{T}$). Thus, in the notation of Theorem 2.22, $A = \frac{1}{9}$ and $B = \frac{5}{9}$ and we see that ψ generates a frame for W_0 with these precise constants.

Moreover, we see that σ_ψ, in this case, is strictly smaller than 1 a. e. (since $\sigma_\psi(\xi) \leq B = \frac{5}{9}$) and $\{\psi_{0k}\}, k \in \mathbb{Z}$ is a frame for W_0, but **not** a tight frame. In this case this system is a Riesz basis as well.

Example 2.30 Let

$$\hat{\varphi}(\xi) = \begin{cases} 1 & \text{if} \quad \xi \in [-\frac{\pi}{4}, \frac{\pi}{4}] \\ -\frac{4}{\pi}\xi + 2 & \text{if} \quad \xi \in [\frac{\pi}{4}, \frac{\pi}{2}] \\ \frac{4}{\pi}\xi + 2 & \text{if} \quad \xi \in [-\frac{\pi}{2}, -\frac{\pi}{4}] \\ 0 & \text{for all other } \xi \end{cases}. \tag{2.31}$$

Then, φ is a pseudo-scaling function. Given this $\hat{\varphi}$, then, the equality (2.12) determines $m(\xi)$ on $[-\frac{\pi}{2}, \frac{\pi}{2})$. Equality (2.11) determines $m(\xi)$ on the remainder of $[-\pi, \pi]$ so that the graph of m on the period interval $[-\pi, \pi]$ is



It is then clear that m is a generalized low pass filter. Thus, φ and m provide us with an MRA TFW by equality (2.13). More precisely,

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\overline{m(\frac{\xi}{2} + \pi)}\hat{\varphi}(\frac{\xi}{2}).$$

Thus, if $\xi \in (\frac{\pi}{4}, \frac{\pi}{2})$, then

$$\sigma_\psi(\xi) = |\hat{\psi}(\xi)|^2 = |m(\frac{\xi}{2} + \pi)|^2 = 1 - |m(\frac{\xi}{2})|^2 = 1 - [\frac{4\xi}{\pi} - 2]^2 > 0.$$

It follows that the left inequality in (2.23) cannot hold for any $A > 0$ since $\sigma_\psi(\frac{\pi}{4}) = 0$ and $\sigma_\psi(\xi)$ is continuous on $[\frac{\pi}{4}, \frac{\pi}{2})$. This gives us an example of an MRA TFW for which $\{\psi_{0k}\}_{k\in\mathbb{Z}}$ is not a frame for W_0.

These two examples show that the question of whether a TFW ψ, or even an MRA TFW, provides us a frame for W_0 when we consider its translates,

$\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$, does not have a simple answer. The situation is much more "structured" if we consider those TFW ψ such that $\{\psi(\cdot - k)\}, k \in \mathbb{Z}$, is a tight frame (of constant 1) for W_0. We call any such ψ a W_0-TF.

Theorem 2.32 *Suppose ψ is a TFW. The following are equivalent: (a) ψ is a W_0-TF, (b) ψ is semi-orthogonal, (c) $\sigma_\psi = \chi_U$ a. e. on $\mathbb{R}$, (d) $||\psi||_2^2 = \sum_{k\in\mathbb{Z}} | < \psi, \psi_{0k} > |^2$.*

Proof: It is clear that (a) implies (d) (this is (2.5) with $f = \psi$ and $H = W_0$). Since $||\psi||_2^2 = \sum_{j,k\in\mathbb{Z}} | < \psi, \psi_{jk} > |^2$ (because $\{\psi_{jk}\}$ is a tight frame), (d) implies $< \psi, \psi_{jk} >= 0$ whenever $j \neq 0$. This clearly implies the semi-orthogonality. Thus, (d) implies (b). If we assume (b), so that $< f, \psi_{jk} >= 0$ whenever $f \in W_0$ and $j \neq 0$, an application of (2.5) gives us

$$||f||_2^2 = \sum_{j,k\in\mathbb{Z}} | < f, \psi_{jk} > |^2 = \sum_{k\in\mathbb{Z}} | < f, \psi_{0k} > |^2$$

and we see that (b) implies (a). Theorem 2.22 with $A = B = 1$ tells us that (a) and (c) are equivalent. ∎

Although the proof of this last theorem is rather simple, the result is not obvious. Some aspects of this theorem are counter-intuitive: the properties (a), (c) and (d) are tied to the inner structure of $W(\psi) = W_0$. On the other hand, (b) provides information about the relationship between W_0 and the other spaces W_j, $j \neq 0$. The assumption that ψ is a TFW is very important. If we do not assume this to be the case, (a) and (c) are still equivalent. That is, $\{\psi(\cdot - k)\}$, $k \in \mathbb{Z}$ is a tight frame for $W(\psi)$ if and only if $\sigma_\psi = \chi_U$. However, (d) is not equivalent to (c). For example, let

$$\hat{\psi}(\xi) = \begin{cases} 2, & 0 \le \xi < \frac{2\pi}{65} \\ \frac{1}{2}, & \frac{2\pi}{65} \le \xi < 2\pi \\ 0, & \text{otherwise} \end{cases},$$

then,

$$||\psi||_2^2 = \frac{1}{2\pi}\int_0^{2\pi} \sigma_\psi(\xi) d\xi = \frac{4}{13},$$

$$\sum_{k\in\mathbb{Z}} | < \psi, \psi_{0k} > |^2 = \frac{1}{2\pi}\int_0^{2\pi} \sigma_\psi^2(\xi) d\xi = \frac{4}{13},$$

but

$$\sigma_\psi(\xi)|_{[0,2\pi)} = 4\chi_{[0,\frac{2\pi}{65})}(\xi) + \frac{1}{4}\chi_{[\frac{2\pi}{65},2\pi)}(\xi).$$

In particular, σ_ψ and χ_U are not equal a. e. and, thus, $\{\psi(\cdot - k)\}$, $k \in \mathbb{Z}$ is not a tight frame for $W(\psi)$. We see, therefore, that (d) is valid, but (c) is not. Theorem 2.32 is useful also for the understanding of several wavelet issues. Let us recall the "four basic equations" of wavelet theory (see Chapter 7 of [HW]). The two most important ones are (2.8) and (2.9), which characterize TFW's. The other two determine the orthonormality of the system $\{\psi_{jk}\}$, $j, k \in \mathbb{Z}$. Using the notation we introduced, the first one is

$$\sigma_\psi(\xi) = 1 \qquad a.\, e. \tag{2.33}$$

and, as we have seen, this characterizes the orthonormality of the system $\{\psi_{0k}\}$, $k \in \mathbb{Z}$. This is just a special case of (c) in Theorem 2.32. The other equation characterizes the semi-orthogonality of our systems and can be written

$$\sum_{k\in\mathbb{Z}} \hat{\psi}(2^j(\xi+2k\pi))\overline{\hat{\psi}(\xi+2k\pi)} = 0 \tag{2.34}$$

a. e. for each $j \geq 1$ (see (1.5) in Chapter 7 of [HW]). Under the assumption of (2.8) and (2.9), (2.33) implies (2.34) (since (2.33) implies $||\psi||_2 = 1$; see Theorem 1.1 of Chapter 7 in [HW]). However, (2.34) does not imply (2.33) since U needs not equal $\mathbb{R}$ a. e. We have seen, however, that (2.34) does imply (c) showing that this last equality (c) is a "proper" generalization of (2.33). It is also of interest to understand how the notion of a Riesz basis fits into our considerations. If a frame is also a (Schauder) basis, then it is a Riesz basis. In the "classical" definition of an MRA in terms of a sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R})$ (see Chaper 2, section 1 of [HW], for example) one usually assumes that the scaling function φ is such that the system $\{\varphi(\cdot - k)\}$, $k \in \mathbb{Z}$, is an orthonormal basis of V_0. It can be shown that if we only assume that this system forms a Riesz basis for V_0, we can then find a scaling function in V_0 whose integral translations form an orthonormal basis for this subspace (see page 48 and 49 of [HW]). Let us describe these facts in the context of the results we have just developed. The assumption that $\{\varphi(\cdot - k)\}$, $k \in \mathbb{Z}$, is a Riesz basis for V_0 means that there exist A and B such that

$$A\sum_{k\in\mathbb{Z}} |\alpha_k|^2 \leq ||\sum_{k\in\mathbb{Z}} \alpha_k\varphi(\cdot - k)||_2^2 \leq B\sum_{k\in\mathbb{Z}} |\alpha_k|^2 \tag{2.35}$$

for all $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. More precisely, it is assumed that each element $f \in V_0$ can be uniquely expressed in the form

$$f = \sum_{k\in\mathbb{Z}} \alpha_k \varphi(\cdot - k)$$

where $\alpha \in \ell^2(\mathbb{Z})$. The inequalities (2.35) imply that in this representation the convergence is in the $L^2(\mathbb{R})$-norm. It is not a priori clear that the coefficients α_k are given by the inner products $< f, \varphi(\cdot - k) >$. We claim that this is the case and $\{\varphi(\cdot - k)\}$, $k \in \mathbb{Z}$, is a frame with the same constants A and B. The arguments we use to obtain (2.18) show that, in the present case, $f \in V_0$ if and only if $\hat{f}(\xi) = t(\xi)\hat{\varphi}(\xi)$ with $t(\xi) = \sum_{k\in\mathbb{Z}} \alpha_k e^{-ik\xi}$ and $||f||_2^2 = \int_{\mathbb{T}} |t(\xi)|^2 \sigma_\varphi(\xi) \frac{d\xi}{2\pi}$. We can choose t to be either χ_E or χ_F as in the end of the proof of Theorem 2.22 to deduce that $0 < A \le \sigma_\varphi(\xi) \le B < \infty$ for a. e. $\xi \in \mathbb{R}$. From this and the arguments used in the proof of Theorem 2.22 we can easily deduce that $\alpha_k =< f, \varphi(\cdot - k) >$. It follows, again by our arguments, that we can then find $\tilde{\varphi} \in V_0$ such that $\{\tilde{\varphi}(\cdot - k)\}$, $k \in \mathbb{Z}$, is a tight frame (of constant 1) for V_0. In this case, however, $U_{\tilde{\varphi}} = \mathbb{R}$ a. e. and, then, $\sigma_{\tilde{\varphi}}(\xi) = 1$ a. e. But this tells us that $\{\tilde{\varphi}(\cdot - k)\}$, $k \in \mathbb{Z}$, is an orthonormal system and, thus, a basis for V_0. All these facts produce a special case of Theorem 2.32 in which $\sigma_\varphi(\xi) = 1$ a. e. All this applies as well to the case where ψ is a TFW and W_0 is the span of $\{\psi_{0k}\}_{k\in\mathbb{Z}}$. In particular, if ψ is a TFW, then, $\{\psi_{0k}\}$, $k \in \mathbb{Z}$, is a Riesz basis for W_0 if and only if there exists A, $0 < A \le 1$, such that $\sigma_\psi(\xi) \ge A$ a. e. (the inequality $\sigma_\psi(\xi) \le 1$ a. e. is automatic —see (2.26) and the comment following). A semi-orthogonal TFW is an orthonormal wavelet if and only if $U_\psi = \mathbb{R}$ a.e. Example 2.27 shows that there exists non-semiorthogonal TFW for which $U_\psi = \mathbb{R}$. Let us present these various levels of orthogonality in the following picture. The picture, Figure 1., indicates also that on each level we can have an MRA TFW as well (so, for the complete understanding of the picture some results from Section 3 are needed).

Finally, we would like to emphasize that Theorem 2.32 and its corollaries show that the periodization function serves as a tool for describing various levels of orthogonality within the class TFWs. Suppose that ψ is a TFW. The periodization function is bounded below on U_ψ if and only if ψ is a W_0-frame (that is, $\{\varphi(\cdot - k)\}$, $k \in \mathbb{Z}$, is a frame for $W_0 = W(\psi)$). The periodization function is bounded below if and only if $\{\psi_{0k} : k \in \mathbb{Z}\}$ is a Riesz basis for $W_0(\psi)$. The periodization function is equal to a characteristic function of a set if and only if ψ is semi-orthogonal. The periodization function is identically one if and only if ψ is an orthonormal wavelet. On the other hand the periodization function is not good in deciding whether a TFW ψ is also an MRA TFW; for example, all orthonormal wavelets (MRA and non-MRA) will have the same periodization function. For this we need to turn our attention to the dimension function.

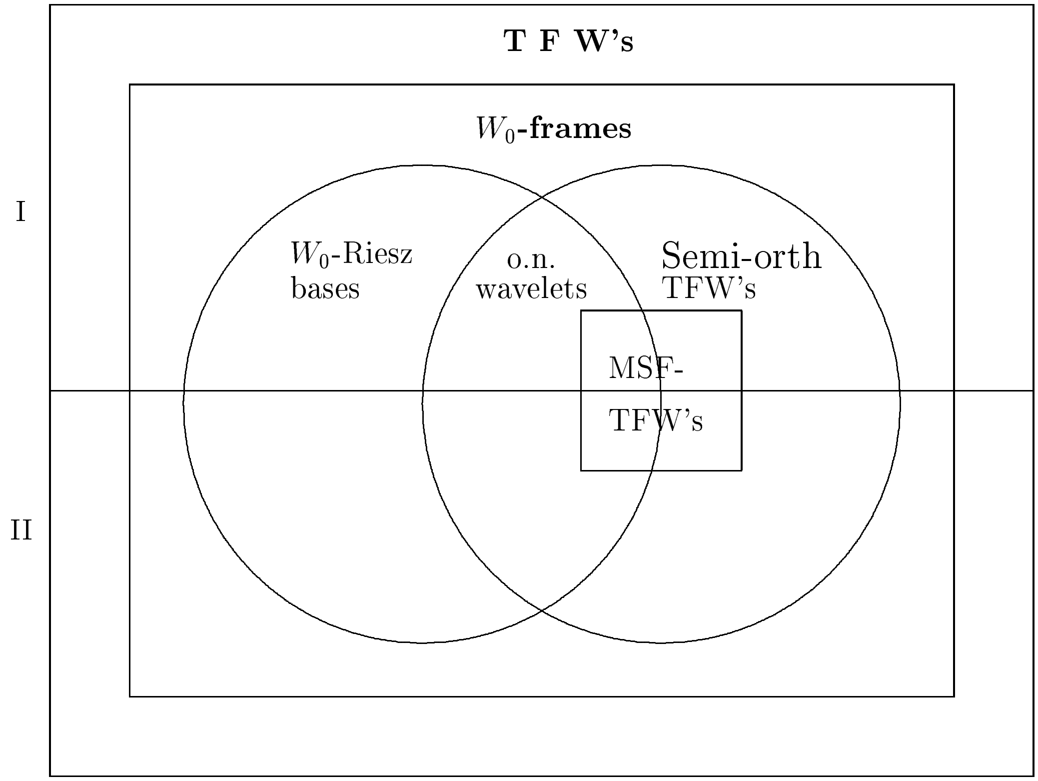


Figure 1: *Various classes of TFWs and their relations. The top part, I, represents the non-MRA TFW's, while the bottom part, II, represents the MRA-TFW's*

3 The Dimension Function

The dimension function of an orthonormal wavelet ψ is integer valued; moreover, unless ψ is an MRA wavelet, it attains each of the integer values in the interval $[0, M]$, where M is the supremum of D_ψ, on sets of positive measure (see [BRS]). We begin our study by showing that these features are also present for the dimension function of a semi-orthogonal TFW.

Theorem 3.1 *Suppose ψ is a TFW. Then ψ is semi-orthogonal if and only* *if D_ψ is integer valued almost everywhere.*

Proof: Let us first assume that D_ψ is integer valued. From (2.3) we see that σ_ψ must also be integer valued a. e. By the observation following (2.26), $\sigma_\psi(\xi) \le 1$ a. e., and the fact that σ_ψ is integer valued, we have $\sigma_\psi(\xi) = \chi_{U_\psi}(\xi)$ a. e. By Theorem 2.32, parts (b) and (c), we conclude that ψ is semiorthogonal. The converse requires a longer argument (this argument is essentially provided in [HW], we just need to adapt it to our situation), we begin by establishing the equality

$$\hat{\psi}(2^n\xi) = \sum_{j=1}^{\infty} \hat{\psi}(2^j\xi) \sum_{k\in\mathbb{Z}} \hat{\psi}(2^n(\xi+2k\pi))\overline{\hat{\psi}(2^j(\xi+2k\pi))} \tag{3.2}$$

a. e. for $n \ge 1$ when ψ is semi-orthogonal. Once this is established we will see that $D_\psi(\xi)$ is, indeed, the dimension function of a linear space. We observe that the sums in (3.2) are absolutely convergent. We do this by applying Schwartz's inequality to the sum with respect to k in order to obtain

$$\begin{aligned} & \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)| \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^n(\xi+2k\pi))| \cdot |\hat{\psi}(2^j(\xi+2k\pi))| \\ \le \; & \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)| (\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^n(\xi+2k\pi))|^2)^{\frac{1}{2}} (\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi+2k\pi))|^2)^{\frac{1}{2}} \\ \le \; & \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)| (\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi+2k\pi))|^2)^{\frac{1}{2}} (\sigma_\psi(2^n\xi))^{\frac{1}{2}}. \end{aligned}$$

Now, we use the fact that $\sigma_\psi(2^n\xi) \le 1$ a. e. and apply Schwartz's inequality to the sum over j in order to majorize the last expression by

$$\begin{aligned} & (\sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2)^{\frac{1}{2}} (\sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi+2k\pi))|^2)^{\frac{1}{2}} \\ & \le \sqrt{D_\psi(\xi)}. \end{aligned}$$

This allows us to interchange the sums in the expression $G_n(\xi)$ which is equal to the right side of (3.2). After adding a term that is equal to 0 a. e. (which is true by (2.34), since $n \geq 1$),

$$\begin{aligned} G_n(\xi) = & \sum_{k\in\mathbb{Z}} \hat{\psi}(2^n(\xi+2k\pi)) \sum_{j=1}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+2k\pi))} \\ & + \hat{\psi}(\xi) \sum_{k\in\mathbb{Z}} \hat{\psi}(2^n(\xi+2k\pi))\overline{\hat{\psi}(\xi+2k\pi)} \\ = & \sum_{k\in\mathbb{Z}} \hat{\psi}(2^n(\xi+2k\pi)) \sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+2k\pi))} \\ = & \sum_{k\in\mathbb{Z}} \hat{\psi}(2^n(\xi+2k\pi)) t_k(\xi) \end{aligned}$$

where $t_k(\xi)$ is defined as in (2.9). Since $t_k(\xi) = 0$ a. e. when k is odd, we obtain

$$\begin{aligned} G_n(\xi) = & \sum_{\ell\in\mathbb{Z}} \hat{\psi}(2^n(\xi+4\ell\pi)) \sum_{j=0}^{\infty} \overline{\hat{\psi}(2^j(\xi+4\ell\pi))}\hat{\psi}(2^j\xi) \\ = & G_{n+1}(\frac{\xi}{2}). \end{aligned}$$

Thus, we have shown that $G_n(\xi) = G_{n-1}(2\xi)$, and consequently, $G_n(\xi) = G_1(2^{n-1}\xi)$ for $n \geq 1$. But

$$\begin{aligned} G_1(\xi) = & \sum_{k\in\mathbb{Z}} \hat{\psi}(2(\xi+2k\pi)) \sum_{j=1}^{\infty} \overline{\hat{\psi}(2^j(\xi+2k\pi))}\hat{\psi}(2^j\xi) \\ = & \sum_{k\in\mathbb{Z}} \hat{\psi}(2\xi+4k\pi) \sum_{j=0}^{\infty} \overline{\hat{\psi}(2^j(2\xi+4k\pi))}\hat{\psi}(2^j\cdot 2\xi) \\ = & \sum_{k\in\mathbb{Z}} \hat{\psi}(2\xi+2k\pi) \sum_{j=0}^{\infty} \overline{\hat{\psi}(2^j(2\xi+2k\pi))}\hat{\psi}(2^j\cdot 2\xi) \\ = & \sum_{j=0}^{\infty} \hat{\psi}(2^j\cdot 2\xi) \cdot \sum_{k\in\mathbb{Z}} \hat{\psi}(2\xi+2k\pi)\overline{\hat{\psi}(2^j(2\xi+2k\pi))} \\ = & \hat{\psi}(2\xi)\sigma_\psi(2\xi) \end{aligned}$$

where the second equality is obtained by writing $\sum_{j=1}^{\infty}$ as a sum of the form $\sum_{j=0}^{\infty}$ and the third equality is derived by adding the terms

$$\hat{\psi}(2\xi + 2(2k+1)\pi)t_{2k+1}(\xi) \quad (= 0 \; a.e.).$$

Finally, the last equality follows by using (2.34) again. This shows that

$$G_n(\xi) = \hat{\psi}(2^n\xi)\sigma_\psi(2^n\xi).$$

Finally, since ψ is a semi-orthogonal TFW, $\sigma_\psi(2^n\xi)$ is either 0 or 1 a.e.; this and the last fact, which implies that $\hat{\psi}(2^n\xi) = 0$ when $\sigma_\psi(2^n\xi) = 0$, gives us

$$G_n(\xi) = \hat{\psi}(2^n\xi) \qquad \text{a.e. for } n \geq 1 \tag{3.3}$$

which concludes the proof of that (3.2) is valid for semi-orthogonal TFW. Now, let

$$\Psi_j(\xi) = \{\hat{\psi}(2^j(\xi + 2k\pi)); k \in \mathbb{Z}\}, \, j \geq 1.$$

Using, again, the fact that $\sigma_\psi(\xi) \leq 1$ we see that $\Psi_j(\xi) \in \ell^2(\mathbb{Z})$ for a.e. $\xi \in \mathbb{R}$. But, from (3.3) we see that

$$\Psi_j(\xi) = \sum_{n=1}^{\infty} < \Psi_n(\xi), \Psi_j(\xi) >_{\ell^2(\mathbb{Z})} \Psi_n(\xi)$$

for $j \geq 1$. By the definition of D_ψ, we have

$$D_\psi(\xi) = \sum_{j=1}^{\infty} ||\Psi_j(\xi)||^2_{\ell^2(\mathbb{Z})} \quad .$$

Hence, the conditions of Lemma 3.7 on page 359 in [HW] are satisfied, i.e. we conclude that

$$D_\psi(\xi) = dim \, \mathbb{F}_\psi(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R} \tag{3.4}$$

where $\mathbb{F}_\psi(\xi) = \overline{span\{\Psi_j(\xi) : j \geq 1\}}$; this is a well-defined subspace of $\ell^2(\mathbb{Z})$. Obviously, (3.4) implies that D_ψ is integer-valued a. e. ■

Notice that the proof of Theorem 3.1 provides us also with the following interesting statement (see also Theorem 3.17):

Corollary 3.5 *Suppose ψ is a TFW. Then, $D_\psi(\xi) = dim\, \mathbb{F}_\psi(\xi)$, for a. e. $\xi \in \mathbb{R}$, if and only if ψ is semi-orthogonal.*

Equation (2.3) is also the cornerstone of the proof of the result by M. Bownik, Z. Rzeszotnik and D. Speegle (for details, see [BRS],[R]) that the dimension function of an orthonormal wavelet which attains value $N > 1$ on a set of positive measure, must also attain value $N - 1$ on a set of positive measure. Equation (2.3) is used there to prove that

$$D_\psi(2\xi) \geq D_\psi(\xi) - 1 \qquad \text{a. e.} \tag{3.6}$$

However, (3.6) is valid for semi-orthogonal TFWs, as well (by (2.3) and the fact that σ_ψ is either 0 or 1). Furthermore, the proof in [R] uses the fact $\int_\pi^\pi D_\psi(\xi) d\xi = 2\pi$ for orthonormal wavelets, in order to contradict it with the implication that $\int_\pi^\pi D_\psi(\xi) d\xi > 2\pi$. This line of argument is obviously valid for semi-orthogonal TFWs, as well since, by (2.2), $\int_\pi^\pi D_\psi(\xi) = ||\hat{\psi}||_2^2 \leq 2\pi$, when ψ is a TFW. Therefore, we obtain the desired result for semi-orthogonal TFWs. More precisely, we have the following result.

Proposition 3.7 *Suppose ψ is a semi-orthogonal TFW. Let $N > 1$ be an integer. If there is a set $A \subseteq \mathbb{R}$ of positive Lebesgue measure, such that*

$D_\psi(\xi) \geq N$ for all $\xi \in A$, then there is a set $B \subseteq \mathbb{R}$ of positive Lebesgue measure such that $D_\psi(\xi) = N - 1$ for all $\xi \in B$.

Remark 3.8 As in the case of orthonormal wavelets, Proposition 3.7 implies that, assuming semi-orthogonality, D_ψ attains all the values between zero and its supremum. The possible difficulty may arise with the value 1. Let us describe precisely what can happen. Suppose ψ is a semi-orthogonal TFW. Hence $D_\psi : \mathbb{R} \to \mathbb{N} \cup \{0\}$. There are two possibilities: either D_ψ attains (on a set of positive measure) some value $N > 1$ or $0 \leq D_\psi \leq 1$ (since it is integer-valued it means D_ψ is either 0 or 1). The second case is possible and we shall see below that it implies that ψ is also an MRA TFW. In that second case there is a slight difference from the case of orthonormal wavelets. Namely, in the case of orthonormal wavelets the condition $D_\psi \leq 1$ would imply $D_\psi \equiv 1$ (a. e.). In the case of semi-orthogonal TFWs we can have D_ψ equal to a characteristic function of a set whose complement also has positive Lebesgue measure. Actually, for a semi-orthogonal TFW ψ the following claim is valid:

$$D_\psi \equiv 1 \text{ if and only if } \psi \text{ is an MRA orthonormal wavelet.} \tag{3.9}$$

Indeed, the "*if* " direction is well-known. Let us establish the "*only if* " part. Suppose $D_\psi \equiv 1$. By (2.4) we have that $\sigma_\psi \equiv 1$. Since $D_\psi \equiv 1$ implies that $||\psi||_2 = 1$ and we also have (2.8) and (2.9), we have that ψ is an orthonormal wavelet. (The MRA part now follows from the "classical" Theorem 1.1). This describes the possibilities in case, $0 \leq D_\psi \leq 1$. Let us turn to the first case, i.e., D_ψ attains some value $N > 1$. By Proposition 3.7, we then have that D_ψ attains all the values $N, N-1, \cdots, 1$. The question is, does it also have to attain value 0 (like in the case of orthonormal wavelets)? The answer is yes; since otherwise, by (2.2), we would have $||\psi||_2 > 1$, which is impossible for TFWs.

There is yet another interesting consequence of Theorem 3.1. Notice that a consequence of (2.8) is that for every TFW ψ we have

$$|\hat{\psi}(\xi)| \leq 1, \qquad a.e. \tag{3.10}$$

An interesting class consists of those TFWs for which $|\hat{\psi}|$ attains only values 0 and 1; in accordance with the orthonormal wavelet terminology we shall call such TFWs, **MSF TFW**s. One may expect that MSF TFWs may or may not be semi-orthogonal. Theorem 3.1, however, implies that they have to lie within the realm of semi-orthogonal TFWs since the corresponding dimension function must be integer-valued. Let us state this in the form of a corollary.

Corollary 3.11 *If ψ is an MSF TFW, then ψ is semiorthogonal.*

We shall turn our attention to the set of MRA TFWs. We begin with an easy part of the theory.

Proposition 3.12 *Suppose ψ is an MRA TFW, and φ is a corresponding pseudo-scaling function (so that φ and ψ are related by (2.13)). Then,*

$$D_\psi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 \tag{3.13}$$

for a. e. $\xi \in \mathbb{R}$, *and*

$$0 \le D_\psi(\xi) \le 1 \qquad a.\, e. \tag{3.14}$$

Proof: Using the definition of D_ψ and (2.16) we immediately obtain (3.13). In order to prove (3.14), let us consider a corresponding generalized low pass filter $m \in \tilde{\mathbf{F}}_\varphi$ (using the notation from [PSWX]). For $n \in \mathbb{N}$ we denote by φ_n the n-th "truncated" partial product obtained from m. That is, $\varphi_n \in L^2(\mathbb{R})$ is defined so that its Fourier transform is

$$\hat{\varphi}_n(\xi) = \chi_{[-2^n\pi, 2^n\pi]}(\xi) \prod_{j=1}^{n} m(2^{-j}\xi). \qquad a.\, e.$$

It is clear that the limit, $|\hat{\varphi}(\xi)|$, of the absolute values of the above "partial products" exists a. e. and the well-known "peeling off" argument (see Lemma 3 in [PSW] or page 370-372 in [HW]) shows that the family $\{\varphi_n(\cdot - k); k \in \mathbb{Z}\}$ forms an orthonormal system. Hence, for every $n \in \mathbb{N}$,

$$\sum_{k \in \mathbb{Z}} |\hat{\varphi}_n(\xi + 2k\pi)|^2 = 1 \quad \text{ for a. e. } \xi \in \mathbb{R}. \tag{3.15}$$

Since the sum over $\mathbb{Z}$ can be considered as an integral with respect to the discrete measure $\sum_{k \in \mathbb{Z}} \delta_k$, we can apply Fatou's lemma, and (3.14) follows from (3.13) and (3.15). ■

Remark 3.16 a) Recall Example (2.27), in particular formula (2.28). It shows that for an MRA TFW the dimension function can oscillate between 0 and 1, attains all the values in between, and be equal to 0 or 1 only on the set of measure 0. b) A natural question is to ask whether the "converse" of (3.14) is valid: suppose that ψ is a TFW and $0 \leq D_\psi \leq 1$ a.e. Does it follow that ψ is an MRA TFW? In this generality, the answer is "no". We have recently learned that D. Speegle has constructed an example of a TFW ψ such that $0 \leq D_\psi \leq 1$ a.e. and $dim\, \mathbb{F}_\psi(\xi) \geq 2$ for ξ in a set of positive Lebesgue measure (recall that $\mathbb{F}_\psi(\xi)$ is introduced in the material preceding (3.4)). Since the example of D. Speegle is going to be published in a separate paper, we shall not discuss it further here. However, we have the complete answer to the question of characterizing MRA TFWs. Althrough D_ψ does not work for all TFWs it provides us with a good condition for semi-orthogonal TFWs; more precisely we have Theorem 3.18 below. This is a consequence of the connection between D_ψ and $\mathbb{F}_\psi$ (established by P. Auscher for orthonormal wavelets) that is still valid for semi-orthogonal TFWs (see proof of Theorem 3.1). Beyond semi-orthogonal TFWs the connection between D_ψ and $\mathbb{F}_\psi$ "breaks down"; as the above mentioned example of D. Speegle shows. Hence, we have a different question to answer: suppose that ψ is a TFW and $dim\, \mathbb{F}_\psi \in \{0, 1\}$ a.e. Does it follow that ψ is an MRA TFW? As we show in Theorem 3.17 below, the answer is positive, and this is actually a desired characterization of MRA TFWs.

We shall complete this section with some of the results we mentioned above. We start by stating a difficult theorem first.

Theorem 3.17 *Suppose ψ is a TFW. Then, ψ is an MRA TFW if and only if $dim\, \mathbb{F}_\psi(\xi) \in \{0, 1\}$, for a.e. $\xi \in \mathbb{R}$.*

Before giving the proof of Theorem 3.17, let us show an important consequence.

Theorem 3.18 *Suppose ψ is a TFW. Then, ψ is a semi-orthogonal MRA TFW if and only if*

$$D_\psi(\xi) \in \{0,1\}, \qquad \textit{for a. e. } \xi \in \mathbb{R}. \tag{3.19}$$

Proof: Suppose that ψ is a semi-orthogonal MRA TFW. Then (3.19) is valid, by (3.14) and Theorem 3.1. Suppose that ψ is a TFW and (3.19) is valid. By Theorem 3.1, (since D_ψ is integer-valued by (3.19)) we conclude that ψ is semi-orthogonal, and, thus, by (3.4), $dim\,\mathbb{F}_\psi(\xi) \in \{0,1\}$, for a. e. $\xi \in \mathbb{R}$. By Theorem 3.17, we conclude that ψ is an MRA TFW. ∎

Remark 3.20 : Theorem 3.18 can be proved directly as well, i. e., without using the full strength of Theorem 3.17.

We shall now turn to the proof of Theorem 3.17. It requires several lemmas.

Proof of Theorem 3.17: We shall prove the easy part first. Suppose that ψ is an MRA TFW; thus, it satisfies (2.13) for an appropriate m and φ. If $dim\,\mathbb{F}_\psi(\xi) \geq 2$, for some $\xi \in \mathbb{R}$, then there exist $k,\ \ell \in \mathbb{Z}$ such that $\Psi_k(\xi)$ and $\Psi_\ell(\xi)$ are linearly independent in $\ell^2(\mathbb{Z})$. It is then impossible to have a vector $v \in \ell^2(\mathbb{Z})$ such that both $\Psi_k(\xi)$ and $\Psi_\ell(\xi)$ are in the one-dimensional subspace $span\{v\}$. However, by (2.13) and (2.12), for every $j \geq 1$, we have

$$\begin{aligned} \hat{\psi}(2^j(\xi+2k\pi)) &= e^{i2^{j-1}(\xi+2k\pi)}\overline{m(2^{j-1}(\xi+2k\pi)+\pi)}\hat{\varphi}(2^{j-1}(\xi+2k\pi)) \\ &= e^{i2^{j-1}\xi}\overline{m(2^{j-1}\xi+\pi)}\prod_{n=1}^{j-2} m(2^n\xi)\hat{\varphi}(\xi+2k\pi), \quad k \in \mathbb{Z}. \end{aligned}$$

Since $\{\hat{\varphi}(\xi + 2k\pi); k \in \mathbb{Z}\}$ is in $\ell^2(\mathbb{Z})$, for a. e. $\xi \in \mathbb{R}$, (by (3.13) and (3.14)) we conclude that $dim\,\mathbb{F}_\psi(\xi)$ is at most 1, for a. e. $\xi \in \mathbb{R}$. We shall now turn our attention to the opposite implication. Its proof is considerably more involved than the last argument. More precisely, we have to prove that ψ is an MRA TFW, assuming that ψ is a TFW such that $dim\,\mathbb{F}_\psi(\xi)$ is either 0 or 1, for a. e. $\xi \in \mathbb{R}$. We remark that $\mathbb{F}_\psi(\xi)$ is always defined for a TFW ψ and is a closed subspaces of $\ell^2(\mathbb{Z})$. This follows from the assumption that ψ is a TFW since this implies that $\sigma_\psi \leq 1$ a. e., which, in turn, implies that, for $j \geq 1$, $\Psi_j(\xi) \in \ell^2(\mathbb{Z})$, for a. e. $\xi \in \mathbb{R}$ (recall the proof of Theorem 3.1). Let us emphasize that we need to prove that there exist a generalized filter m and a corresponding pseudo-scaling function φ such that (2.13) is satisfied, i. e., for a. e. $\xi \in \mathbb{R}$

$$\hat{\psi}(2\xi) = e^{i\xi}\overline{m(\xi + \pi)}\hat{\varphi}(\xi). \tag{3.21}$$

We begin by proving that it is enough to show that there exist a generalized filter m_0, a corresponding pseudo-scaling function φ_0, and a 2π-periodic, unimodular function μ such that

$$\hat{\psi}(2\xi) = e^{i\xi}\mu(2\xi)\overline{m_0(\xi + \pi)}\hat{\varphi}_0(\xi) \quad a.\,e. \tag{3.22}$$

Indeed, if (3.22) is valid, then we can apply Lemma 1 from [WUTAM] to μ, and conclude that there exists a 2π-periodic, unimodular function t such that

$$\mu(2\xi) = \overline{t(2\xi)}t(\xi)t(\xi + \pi) \quad a.\,e. \tag{3.23}$$

We then define m by

$$m(\xi) := t(2\xi)\overline{t(\xi)}m_0(\xi),$$

and $\varphi \in L^2(\mathbb{R})$ by

$$\hat{\varphi}(\xi) := t(\xi)\hat{\varphi}_0(\xi).$$

It is obvious that the 2π-periodicity of t and m_0 implies the 2π-periodicity of m, while the unimodularity of t implies that $|m| = |m_0|$. It follows that m is a generalized filter. The following calculation shows that φ is the corresponding pseudo-scaling function:

$$\begin{aligned}\hat{\varphi}(2\xi) &= t(2\xi)\hat{\varphi}_0(2\xi) = t(2\xi)m_0(\xi)\hat{\varphi}_0(\xi) = \\ &= t(2\xi)m_0(\xi)\overline{t(\xi)}t(\xi)\hat{\varphi}_0(\xi) = m(\xi)\hat{\varphi}(\xi).\end{aligned}$$

Similarly, it is easy to check (3.21):

$$\begin{aligned}e^{i\xi}\overline{m(\xi+\pi)}\hat{\varphi}(\xi) &= e^{i\xi}\overline{t(2\xi+2\pi)}t(\xi+\pi)\overline{m_0(\xi+\pi)}t(\xi)\hat{\varphi}_0(\xi) \\ &= e^{i\xi}\mu(2\xi)\overline{m_0(\xi+\pi)}\hat{\varphi}_0(\xi) = \hat{\psi}(2\xi).\end{aligned}$$

In the last calculation we have used (3.23) and (3.22). Hence, we have shown that it is enough to establish (3.22) in order to complete the proof of the theorem. We begin our construction of m_0 and φ_0 by considering first the following sets:

$$\mathcal{Z} := \{\xi \in \mathbb{R} : D_\psi(\xi) = 0\} \tag{3.24}$$

and, for $j \in \mathbb{N}$, we define the set $\mathcal{P}_j$ to be

$$\{\xi \in \mathbb{R} : ||\Psi_j(\xi)||_{\ell^2} \neq 0 \text{ and } ||\Psi_\ell(\xi)||_2 = 0 \text{ for } 1 \leq \ell \leq j-1\}. \tag{3.25}$$

It is easy to see that $\mathcal{Z}$ is the set where all vectors $\Psi_j(\xi)$, $j \geq 1$, are zero (or, equivalently, where $dim\, \mathbb{F}_\psi(\xi) = 0$). Moreover, the sets $\mathcal{Z}$ and the $\mathcal{P}_j, s$

are 2π-periodic, measurable and they form a partition of $\mathbb{R}$. We shall now define a function of $\xi \in \mathbb{R}$ and denote it by $\hat{\varphi}_0$, although, at first, we do not know that it belongs to $L^2(\mathbb{R})$:

$$\hat{\varphi}_0(\xi) := \begin{cases} 0, & \xi \in \mathcal{Z} \\ \sqrt{\dfrac{D_\psi(\xi)}{\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi+2k\pi))|^2}} \cdot \hat{\psi}(2^j\xi), & \xi \in \mathcal{P}_j. \end{cases} \tag{3.26}$$

Observe that, by the definition of the $\mathcal{P}_j's$, (3.26) makes sense and defines a measurable function $\hat{\varphi}_0 : \mathbb{R} \to \mathbb{C}$. Furthermore, this function clearly satisfies

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}_0(\xi + 2k\pi)|^2 = D_\psi(\xi) \quad a.e. \tag{3.27}$$

Using (2.2) and (3.27) we obtain that $||\hat{\varphi}_0||_2^2 = ||\hat{\psi}||_2^2$. It follows that $\hat{\varphi}_0 \in L^2(\mathbb{R})$ and the notation makes sense; meaning that there is a $\varphi_0 \in L^2(\mathbb{R})$ such that $\hat{\varphi}_0$ is given by (3.26). We now consider properties that are more difficult to establish.

Lemma 3.28 *For a. e. $\xi \in \mathbb{R}$ we have,*

$$|\hat{\varphi}_0(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2. \tag{3.29}$$

Proof of Lemma 3.28: For $\xi \in \mathcal{Z}$, (3.29) is trivially true. Consider $\xi \in \mathcal{P}_\ell$, $\ell \in \mathbb{N}$. Hence, $\Psi_\ell(\xi) \neq 0$ and, by our assumption, $dim\,\mathbb{F}_\psi(\xi) = 1$. This implies that for every $j \geq 1$ there exists a 2π-periodic, measurable function $\lambda_j^\ell : \mathcal{P}_\ell \to \mathbb{C}$, such that, for a. e. $\xi \in \mathcal{P}_\ell$,

$$\Psi_j(\xi) = \lambda_j^\ell(\xi)\Psi_\ell(\xi). \tag{3.30}$$

Coordinatewise, this means that for every $k \in \mathbb{Z}$,

$$\hat{\psi}(2^j(\xi + 2k\pi)) = \lambda_j^\ell(\xi)\hat{\psi}(2^\ell(\xi + 2k\pi)), \tag{3.31}$$

for a. e. $\xi \in \mathcal{P}_\ell$. It follows that the right hand side of (3.29) satisfies

$$\sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2 = \left[\sum_{j=1}^{\infty} |\lambda_j^\ell(\xi)|^2\right] \cdot |\hat{\psi}(2^\ell\xi)|^2 \tag{3.32}$$

for a. e. $\xi \in \mathcal{P}_\ell$. Keeping in mind formula (3.26) for $\xi \in \mathcal{P}_\ell$, (3.32) implies that it is enough to prove

$$\sum_{j=1}^{\infty} |\lambda_j^\ell(\xi)|^2 = \frac{D_\psi(\xi)}{\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2}, \tag{3.33}$$

for a. e. $\xi \in \mathcal{P}_\ell$. The following calculation is a consequence of (3.31), and it proves (3.33). For a. e. $\xi \in \mathcal{P}_\ell$ we obtain

$$\begin{aligned} D_\psi(\xi) &= \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 \\ &= \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\lambda_j^\ell(\xi)|^2 |\hat{\psi}(2^\ell(\xi + 2k\pi))|^2 \\ &= \left[\sum_{j=1}^{\infty} |\lambda_j^\ell(\xi)|^2\right] \cdot \left[\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^\ell(\xi + 2k\pi))|^2\right]. \end{aligned}$$

This completes the proof of Lemma 3.28, since $\mathcal{Z}$ and $\mathcal{P}_j's$ cover all of $\mathbb{R}$. ■

Lemma 3.34 *There exists a 2π-periodic, measurable function m_1: $\mathcal{Z}^c \to \mathbb{C}$ such that*

$$|m_1(\xi)| \le 1, \quad \textit{for a. e. } \xi \in \mathcal{Z}^c, \tag{3.35}$$

and

$$\hat{\psi}(2\xi) = e^{i\xi}\, \overline{m_1(\xi)}\, \hat{\varphi}_0(\xi), \ \textit{for a. e. } \xi \in \mathcal{Z}^c. \tag{3.36}$$

Proof of Lemma 3.34: Notice that the 2π-periodicity of m_1 is consistent with the 2π-periodicity of the measurable set

$$\mathcal{Z}^c = \bigcup_{\ell \in \mathbb{N}} \mathcal{P}_\ell. \tag{3.37}$$

We define m_1 by the following formula

$$m_1(\xi) := \begin{cases} 0, & \xi \in \mathcal{P}_\ell, \ell \geq 2 \\ e^{i\xi}\sqrt{\dfrac{\sum_{k\in\mathbb{Z}} |\hat{\psi}(2(\xi + 2k\pi))|^2}{D_\psi(\xi)}}, & \xi \in \mathcal{P}_1. \end{cases} \tag{3.38}$$

Since all the functions and sets involved in (3.38) are measurable and 2π-periodic, it is obvious that (3.38) defines a 2π-periodic, measurable function $m_1 : \mathcal{Z}^c \to \mathbb{C}$. Furthermore, $\sum_{k\in\mathbb{Z}} |\hat{\psi}(2(\xi + 2k\pi))|^2 \leq D_\psi(\xi)$, implies that (3.35) is valid. For $\xi \in \mathcal{P}_1$, (3.36) is the direct consequence of (3.26) and (3.38). For $\xi \in \mathcal{P}_\ell, \ell \geq 2$, $\Psi_1(\xi) = 0$. In particular, $\hat{\psi}(2\xi) = 0$; thus (3.36) is trivially satisfied. ■

Lemma 3.39 *There exists a 2π-periodic, measurable function $m_0 : \mathcal{Z}^c \to \mathbb{C}$ such that*

$$|m_0(\xi)| \leq 1, \; \textit{for a. e. } \xi \in \mathcal{Z}^c, \tag{3.40}$$

and

$$\hat{\varphi}_0(2\xi) = m_0(\xi)\hat{\varphi}_0(\xi), \; \textit{for a. e. } \xi \in \mathcal{Z}^c. \tag{3.41}$$

Proof of Lemma 3.39: Again, by (3.37), the 2π-periodicity will be clear from the argument, and we need only to prove (3.40) and (3.41) on $\bigcup_{\ell\in\mathbb{N}} \mathcal{P}_\ell$.

Consider $\mathcal{P}_\ell, \ell \geq 2$, first. For a. e. $\xi \in \mathcal{P}_\ell$ we obtain

$$
\begin{aligned}
&||\Psi_{\ell-1}(2\xi)||^2_{\ell^2(\mathbb{Z})} = \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^{\ell-1}(2\xi + 2k\pi))|^2 \\
= \quad & \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^\ell(\xi + k\pi))|^2 \geq \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^\ell(\xi + 2k\pi))|^2 \\
= \quad & ||\Psi_\ell(\xi)||^2_{\ell^2(\mathbb{Z})} > 0.
\end{aligned}
$$

This and the definition (3.25) of the $\mathcal{P}_j' s$ imply that for a. e. $\xi \in \mathcal{P}_\ell$, there exists $k = k(\xi) \in \{1, \cdots, \ell - 1\}$ such that $2\xi \in \mathcal{P}_k$. It is easy to check, using (3.25), that $\xi \to k(\xi)$ is 2π-periodic and measurable (defined on $\mathcal{P}_\ell$). Since, by our assumption, $dim\,\mathbb{F}_\psi(2\xi) = 1$ for a. e. $\xi \in \mathcal{P}_\ell$, there exists a 2π-peridoic, measurable function $\lambda : \mathcal{P}_\ell \to \mathbb{C}$ such that, for a. e. $\xi \in \mathcal{P}_\ell$,

$$\Psi_{\ell-1}(2\xi) = \lambda(\xi)\Psi_{k(\xi)}(2\xi). \tag{3.42}$$

Notice that $\lambda(\xi) \neq 0$, since otherwise we would have

$$0 = ||\Psi_{\ell-1}(2\xi)||^2_{\ell^2(\mathbb{Z})} \geq ||\Psi_\ell(\xi)||^2_{\ell^2(\mathbb{Z})} \geq 0,$$

which would imply $\Psi_\ell(\xi) = 0$; this is impossible for $\xi \in \mathcal{P}_\ell$. Hence, (3.42) implies that, for a. e. $\xi \in \mathcal{P}_\ell$,

$$
\begin{aligned}
\hat{\varphi}_0(2\xi) &= \sqrt{\frac{D_\psi(2\xi)}{\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j(2\xi + 2j\pi))|^2}} \cdot \hat{\psi}(2^k \cdot 2\xi) \\
&= \sqrt{\frac{D_\psi(2\xi)}{\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j(2\xi + 2j\pi))|^2}} \cdot \frac{1}{\lambda(\xi)}\hat{\psi}(2^{\ell-1} \cdot 2\xi). \qquad (3.43)
\end{aligned}
$$

Notice that (3.43) shows that there exists a 2π-periodic, measurable function $A : \mathcal{P}_\ell \to \mathbb{C}$ such that, for a. e. $\xi \in \mathcal{P}_\ell$,

$$\hat{\varphi}_0(2\xi) = A(\xi)\hat{\psi}(2^\ell\xi). \tag{3.44}$$

At the same time it follows directly from (3.26), that there exists a 2π-periodic, measurable function $B : \mathcal{P}_\ell \to \mathbb{C}$, such that, for a. e. $\xi \in \mathcal{P}_\ell$, $B(\xi) \neq 0$ and

$$\hat{\varphi}_0(\xi) = B(\xi)\hat{\psi}(2^\ell \xi). \tag{3.45}$$

We define m_0 on $\mathcal{P}_\ell$ by

$$m_0(\xi) := \frac{A(\xi)}{B(\xi)}. \tag{3.46}$$

By (3.44) and (3.45) it is clear that m_0 is 2π-periodic on $\mathcal{P}_\ell$, measurable and satifies (3.41) on $\mathcal{P}_\ell$. Therefore, it remains to define m_0 on $\mathcal{P}_1$. Observe that for $\xi \in \mathcal{P}_1$, either $2\xi \in \mathcal{Z}$, or, otherwise, there is a measurable and 2π-periodic function $\xi \to \ell(2\xi) \in \mathbb{N}$, such that $2\xi \in \mathcal{P}_{\ell(2\xi)}$. If $2\xi \in \mathcal{Z}$, then we define $m_0(\xi)$ to be 0. Otherwise, we obtain, by (3.26), that there exists a 2π-periodic, measurable function $\tilde{A}$ such that

$$\hat{\varphi}_0(2\xi) = \tilde{A}(2\xi)\hat{\psi}(2^{\ell(2\xi)} \cdot 2\xi) = \tilde{A}(2\xi)\hat{\psi}(2^{\ell(2\xi)+1}\xi).$$

On the other hand, since $\xi \in \mathcal{P}_1$, we know that there exists a 2π-periodic, measurable function $\tilde{\lambda}$, such that

$$\Psi_{\ell(2\xi)+1}(\xi) = \tilde{\lambda}(\xi) \cdot \Psi_1(\xi),$$

and a 2π-periodic, measurable function $\tilde{B} \neq 0$, such that

$$\hat{\varphi}_0(\xi) = \tilde{B}(\xi) \cdot \hat{\psi}(2\xi).$$

Hence, for $\xi \in \mathcal{P}_1$ and $2\xi \notin \mathcal{Z}$, we define $m_0(\xi)$ by

$$m_0(\xi) := \frac{\tilde{A}(2\xi)\tilde{\lambda}(\xi)}{\tilde{B}(\xi)}. \tag{3.47}$$

As before, it is now clear that m_0 is 2π-periodic, measurable on $\mathcal{P}_1$ and satisfies (3.41) on $\mathcal{P}_1$. It follows that we have a 2π-periodic, measurable function $m_0 : \mathcal{Z}^c \to \mathbb{C}$, such that (3.41) is satisfied. Let us prove (3.40). For $\xi \in \mathcal{Z}^c$, we have, by (3.27), that there exists $k \in \mathbb{Z}$ such that $\hat{\varphi}_0(\xi+2k\pi) \neq 0$. Since m_0 is 2π-periodic we obtain, by (3.41),

$$\begin{aligned} \hat{\varphi}_0(2(\xi + 2k\pi)) &= m_0(\xi + 2k\pi)\hat{\varphi}_0(\xi + 2k\pi) \\ &= m_0(\xi)\hat{\varphi}_0(\xi + 2k\pi). \end{aligned}$$

Hence,

$$|m_0(\xi)| = \frac{|\hat{\varphi}_0(2(\xi + 2k\pi))|}{|\hat{\varphi}_0(\xi + 2k\pi)|}.$$

However, Lemma 3.28 implies that for a. e. $u \in \mathbb{R}$, $|\hat{\varphi}_0(2u)|^2 \leq |\hat{\varphi}_0(u)|^2$. Thus, $|m_0(\xi)| \leq 1$, and this completes the proof of Lemma 3.39. ∎

Let us extend the definitions of m_0 and m_1 to $\mathcal{Z}$, as well. Since m_0 is already defined on $\mathcal{Z}^c$, the following definition of m_1 on $\mathcal{Z}$ makes sense:

$$m_1(\xi) := \begin{cases} \dfrac{1}{\sqrt{2}}, & \xi \in \mathcal{Z}, \quad \xi + \pi \in \mathcal{Z} \\ m_0(\xi + \pi), & \xi \in \mathcal{Z}, \quad \xi + \pi \notin \mathcal{Z}. \end{cases} \tag{3.48}$$

Using Lemma 3.34, Lemma 3.39 and (3.48), we conclude that $m_1 : \mathbb{R} \to \mathbb{C}$ is a 2π-periodic, measurable function, such that

$$|m_1(\xi)| \leq 1, \text{ for a. e. } \xi \in \mathbb{R}. \tag{3.49}$$

Furthermore, we claim that (3.36) is now satisfied for a. e. $\xi \in \mathbb{R}$. By Lemma 3.34, we need to check (3.36) only on $\mathcal{Z}$. However, for $\xi \in \mathcal{Z}$, we have that $\hat{\varphi}_0(\xi) = 0$, by (3.26), and $\hat{\psi}(2\xi) = 0$, since $D_\psi(\xi) = 0$. Hence, on $\mathcal{Z}$ (3.36) is satisfied irrespective of the value of m_1. We have established, therefore, that

$$\hat{\psi}(2\xi) = e^{i\xi}\,\overline{m_1(\xi)}\,\hat{\varphi}_0(\xi), \text{ for a.e. } \xi \in \mathbb{R}. \tag{3.50}$$

Let us turn our attention to m_0. Now that m_1 is defined on all of $\mathbb{R}$, we define m_0 on $\mathcal{Z}$ by

$$m_0(\xi) := m_1(\xi + \pi), \text{ for } \xi \in \mathcal{Z}. \tag{3.51}$$

Obviously, $m_0 : \mathbb{R} \to \mathbb{C}$ is a 2π-periodic, measurable function such that

$$|m_0(\xi)| \leq 1, \text{ for a.e. } \xi \in \mathbb{R}. \tag{3.52}$$

Again, we claim that such m_0 satisfies (3.41) on $\mathbb{R}$. And, again, because of Lemma 3.39, it is enough to check (3.41) on $\mathcal{Z}$. Since $\hat{\varphi}_0(\xi) = 0$, for $\xi \in \mathcal{Z}$, it is enough to show that $\hat{\varphi}_0(2\xi) = 0$, for $\xi \in \mathcal{Z}$. but this is an immediate consequence of Lemma 3.28. We conclude that, for a.e. $\xi \in \mathcal{R}$,

$$\hat{\varphi}_0(2\xi) = m_0(\xi)\,\hat{\varphi}_0(\xi). \tag{3.53}$$

The following lemma connects m_0 and m_1:

Lemma 3.54 *For a. e. $\xi \in \mathbb{R}$,*

$$|m_0(\xi)|^2 + |m_1(\xi)|^2 = 1. \tag{3.55}$$

Proof of Lemma 3.54: Consider $\xi \in \mathcal{Z}$ first. If $\xi + \pi \in \mathcal{Z}$, then, by (3.48) and (3.51), $m_1(\xi) = \dfrac{1}{\sqrt{2}}$, and, since $(\xi + \pi) + \pi = \xi + 2\pi \in \mathcal{Z}$,

$$m_0(\xi) = m_1(\xi + \pi) = \frac{1}{\sqrt{2}}.$$

Obviously, (3.55) is satisfied. If $\xi \in \mathcal{Z}$, and $\xi + \pi \notin \mathcal{Z}$, then, by (3.48) and (3.51),

$$|m_0(\xi)|^2 + |m_1(\xi)|^2 = |m_1(\xi+\pi)|^2 + |m_0(\xi+\pi)|^2.$$

Hence, (3.55) is going to be satisfied if we can prove it on $\mathcal{Z}^c$. We present this argument (notice that this would also complete the proof of this lemma): Lemma 3.28 implies that, for a. e. $\xi \in \mathbb{R}$,

$$|\hat{\varphi}_0(\xi)|^2 = |\hat{\psi}(2\xi)|^2 + |\hat{\varphi}_0(2\xi)|^2. \tag{3.56}$$

¿From (3.50) and (3.53) we obtain that, for a. e. $\xi \in \mathbb{R}$,

$$|\hat{\varphi}_0(\xi)|^2 = \left[|m_1(\xi)|^2 + |m_0(\xi)|^2\right] \cdot |\hat{\varphi}_0(\xi)|^2. \tag{3.57}$$

Using the fact that m_1 and m_0 are 2π-periodic we can periodize (3.57). This periodization and (3.27) provides us with

$$D_\psi(\xi) = \left[|m_1(\xi)|^2 + |m_0(\xi)|^2\right] \cdot D_\psi(\xi) \quad \text{a. e.} \tag{3.58}$$

Recall that on $\mathcal{Z}^c$, $D_\psi(\xi) \neq 0$; thus (3.58) provides us with (3.55) on $\mathcal{Z}^c$. ■ The following lemma establishes that m_0 and m_1 are generalized filters, and provides the crucial step to find μ such that (3.22) is satisfied. It is interesting to observe that the proof of this lemma relies on the fact that $t_q(\xi) = 0$, for $q \in \mathbb{Z} \setminus 2\mathbb{Z}$ (recall the definition of $t_q(\xi)$ in (2.9)).

Lemma 3.59 *For a. e. $\xi \in \mathbb{R}$,*

$$m_0(\xi)\,\overline{m_0(\xi+\pi)} = m_1(\xi+\pi)\,\overline{m_1(\xi)}, \tag{3.60}$$

and

$$|m_0(\xi)| = |m_1(\xi + \pi)|, \tag{3.61}$$

and

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1 = |m_1(\xi)|^2 + |m_1(\xi + \pi)|^2. \tag{3.62}$$

Proof of Lemma 3.59: Let us first observe that we only need to prove (3.60). Indeed, (3.60) and Lemma 3.54 imply that

$$\begin{aligned} & |m_0(\xi)|^2 = |m_0(\xi)|^2 \cdot |m_0(\xi + \pi)|^2 + |m_0(\xi)|^2 \cdot |m_1(\xi + \pi)|^2 \\ = \quad & |m_1(\xi + \pi)|^2 \cdot |m_1(\xi)|^2 + |m_0(\xi)|^2 \cdot |m_1(\xi + \pi)|^2 = |m_1(\xi + \pi)|^2, \end{aligned}$$

this clearly implies (3.61). Lemma 3.54 and (3.61) imply (3.62) straightforwardly. Let us prove (3.60). Consider $\xi \in \mathcal{Z} \cup (\mathcal{Z} + \pi)$ first. We have three possibilities: $\xi \in \mathcal{Z}$ and $\xi + \pi \notin \mathcal{Z}$, $\xi \in \mathcal{Z}$ and $\xi + \pi \in \mathcal{Z}$, $\xi \notin \mathcal{Z}$ and $\xi + \pi \in \mathcal{Z}$. In the case $\xi \in \mathcal{Z}$ and $\xi + \pi \notin \mathcal{Z}$, we have, by (3.48) and (3.51),

$$m_0(\xi) = m_1(\xi + \pi), m_1(\xi) = m_0(\xi + \pi);$$

which clearly establishes (3.60). In the case $\xi \in \mathcal{Z}$ and $\xi + \pi \in \mathcal{Z}$, we have, by (3.48) and (3.51),

$$m_0(\xi) = m_1(\xi + \pi), m_1(\xi) = \frac{1}{\sqrt{2}} = m_0(\xi + \pi).$$

Again, (3.60) follows. Finally, if $\xi \notin \mathcal{Z}$ and $\xi + \pi \in \mathcal{Z}$, then $(\xi + \pi) + \pi \notin \mathcal{Z}$, so we have, by (3.48) and (3.51),

$$m_1(\xi + \pi) = m_0(\xi + 2\pi) = m_0(\xi), m_0(\xi + \pi) = m_1(\xi).$$

Obviously, (3.60) follows. Let us consider $\xi \in [\mathcal{Z} \cup (\mathcal{Z} + \pi)]^c$. We shall apply the following calculation. Using (3.50), (3.53) and (3.55), we obtain, for $j \geq 1$, $q \in \mathbb{Z} \setminus 2\mathbb{Z}$, and a. e. $u \in \mathbb{R}$,

$$
\begin{aligned}
&\hat{\psi}(2^j u)\,\overline{\hat{\psi}(2^j(u+2q\pi))}\\
&= e^{i2^{j-1}u}\,\overline{m_1(2^{j-1}u)}\,\hat{\varphi}_0(2^{j-1}u)\,\overline{e^{i2^{j-1}(u+2q\pi)}}\,m_1(2^{j-1}(u+2q\pi))\,\overline{\hat{\varphi}_0(2^{j-1}(u+2q\pi))}\\
&= |m_1(2^{j-1}u)|^2\,\hat{\varphi}_0(2^{j-1}u)\,\overline{\hat{\varphi}_0(2^{j-1}(u+2q\pi))}\\
&= \left[1-|m_0(2^{j-1}u)|^2\right]\,\hat{\varphi}_0(2^{j-1}u)\,\overline{\hat{\varphi}_0(2^{j-1}(u+2q\pi))}\\
&= \hat{\varphi}_0(2^{j-1}u)\,\overline{\hat{\varphi}_0(2^{j-1}(u+2q\pi))} - \hat{\varphi}_0(2^j u)\,\overline{\hat{\varphi}_0(2^j(u+2q\pi))}.
\end{aligned}
$$

We shall now apply Lemma 2.14 from [PSWX] on $\hat{\varphi}_0(\cdot)\overline{\hat{\varphi}_0(\cdot+2q\pi)}$, and (2.9), to obtain, for $q \in \mathbb{Z} \setminus 2\mathbb{Z}$ and a. e. $u \in \mathbb{R}$,

$$
\begin{aligned}
0 \quad &= t_q(u) = \sum_{j=0}^{\infty} \hat{\psi}(2^j u)\,\overline{\hat{\psi}(2^j(u+2q\pi))}\\
&= \hat{\psi}(u)\,\overline{\hat{\psi}(u+2q\pi)} + \hat{\varphi}_0(u)\,\overline{\hat{\varphi}_0(u+2q\pi)}\\
&= e^{i\frac{u}{2}}\,\overline{m_1(\frac{u}{2})}\,\hat{\varphi}_0(\frac{u}{2})\cdot \overline{e^{i(\frac{u}{2}+q\pi)}}\,m_1(\frac{u}{2}+q\pi)\,\overline{\hat{\varphi}_0(\frac{u}{2}+q\pi)}\\
&\quad + m_0(\frac{u}{2})\,\hat{\varphi}_0(\frac{u}{2})\cdot\overline{m_0(\frac{u}{2}+q\pi)}\,\overline{\hat{\varphi}_0(\frac{u}{2}+q\pi)}\\
&= \left[m_0(\frac{u}{2})\,\overline{m_0(\frac{u}{2}+q\pi)} - \overline{m_1(\frac{u}{2})}\,m_1(\frac{u}{2}+q\pi)\right]\cdot\hat{\varphi}_0(\frac{u}{2})\,\hat{\varphi}_0(\frac{u}{2}+q\pi).
\end{aligned}
$$

Suppose now $\xi \notin \mathcal{Z}$ and $\xi + \pi \notin \mathcal{Z}$. By (3.27) there exist $k, \ell \in \mathbb{Z}$ such that

$$\hat{\varphi}_0(\xi + 2k\pi) \neq 0 \neq \hat{\varphi}_0(\xi + (2\ell+1)\pi).$$

This implies that

$$\hat{\varphi}_0(\xi+2k\pi)\cdot\overline{\hat{\varphi}_0(\xi+2k\pi+[2(\ell-k)+1]\pi)} \neq 0.$$

Take in the above computation

$$\frac{u}{2} = \xi + 2k\pi \text{ and } q = 2(\ell - k) + 1,$$

and we conclude

$$\begin{aligned} 0 &= m_0(\xi + 2k\pi)\overline{m_0(\xi + \pi + 2\ell\pi)} - \overline{m_1(\xi + 2k\pi)}m_1(\xi + \pi + 2\ell\pi) \\ &= m_0(\xi)\overline{m_0(\xi + \pi)} - \overline{m_1(\xi)}m_1(\xi + \pi). \end{aligned}$$

This proves the case $\xi \in [\mathcal{Z} \cup (\mathcal{Z} + \pi)]^c$, and completes the proof of this lemma. ∎

We have established so far that m_0 and m_1 are generalized filters, that φ_0 is a pseudo-scaling function and that m_0 is its corresponding filter, and that $\hat{\psi}$ satisfies (3.50). We shall now prove that (3.50) provides us with (3.22), which is going to complete the proof of the theorem. To this end we consider sets H and H^c, where H is defined by

$$H := \{\xi \in \mathbb{R} : m_0(\xi + \pi) = 0\}. \tag{3.63}$$

Clearly, both H and H^c are measurable and 2π-periodic. Furthermore, by (3.62), we have

$$\text{if } \xi \in H, \text{ then } \xi + \pi \in H^c \quad a.e. \tag{3.64}$$

We shall define a function s in the following way. For $\xi \in H^c$, we define s by

$$s(\xi) := \frac{m_1(\xi)}{m_0(\xi + \pi)}. \tag{3.65}$$

By (3.61) it follows that $s : H^c \to \mathbb{C}$ is a 2π-periodic, unimodular (and, hence, measurable) function. For $\xi \in H$, we use (3.64) to conclude that $\xi + \pi \in H^c$, and we define s by

$$s(\xi) := s(\xi + \pi). \tag{3.66}$$

Hence, s is a unimodular, 2π-periodic function on $\mathbb{R}$, which satisfies

$$m_1(\xi) = s(\xi) m_0(\xi + \pi) \quad \text{a. e.} \tag{3.67}$$

We claim that s is π-periodic. Let us consider $\xi \in \mathbb{R}$ such that $m_0(\xi) \cdot \overline{m_0(\xi + \pi)} = 0$. By (3.64) we have in this case that either ($\xi \in H$ and $\xi + \pi \notin H$) or ($\xi \notin H$ and $\xi + \pi \in H$). In both cases (3.66) and 2π-periodicity of s provides us with the conclusion that $s(\xi) = s(\xi + \pi)$. Let us consider $\xi \in \mathbb{R}$ such that $m_0(\xi) \cdot \overline{m_0(\xi + \pi)} \neq 0$. In this case we apply (3.60), (3.67) and the unimodularity of s, to obtain

$$\begin{aligned} m_0(\xi)\overline{m_0(\xi + \pi)} &= m_1(\xi + \pi)\overline{m_1(\xi)} \\ &= s(\xi + \pi) m_0(\xi + 2\pi) \overline{s(\xi) m_0(\xi + \pi)} \\ &= [s(\xi + \pi)\overline{s(\xi)}][m_0(\xi)\overline{m_0(\xi + \pi)}]. \end{aligned}$$

It follows that $s(\xi) = s(\xi + \pi)$. We define $\mu : \mathbb{R} \to \mathbb{C}$ by

$$\mu(\xi) := \overline{s(\frac{\xi}{2})}, \quad \xi \in \mathbb{R}. \tag{3.68}$$

Since s is π-periodic, we conclude that μ is a 2π-periodic, unimodular function. Hence, m_0, φ_0, and μ satisfy all the necessary requirements. It remains to check that they satisfy (3.22), as well. Indeed, by (3.50), (3.67) and (3.68),

we obtain
$$\begin{aligned}\hat{\psi}(2\xi) &= e^{i\xi}\overline{m_1(\xi)}\hat{\varphi}_0(\xi)\\ &= e^{i\xi}\overline{s(\xi)m_0(\xi+\pi)}\hat{\varphi}_0(\xi)\\ &= e^{i\xi}\mu(2\xi)\overline{m_0(\xi+\pi)}\hat{\varphi}_0(\xi) \quad a.e.\end{aligned}$$
This concludes the proof of Theorem 3.17. ■

4 Connectivity

In this section, we shall establish the path-connectivity of the set of all MSF TFWs. This parallels a result of D. Speegle [S]. Let us recall from Section 3 that a TFW ψ is called an MSF TFW if

$$|\hat{\psi}(\xi)| = \chi_\omega(\xi) \tag{4.1}$$

for some subset ω of $\mathbb{R}$. Such an ω is called a TFW-set. That is, ω is a TFW-set if there exists a TFW ψ such that (4.1) holds. TFW sets ω are characterized by the following two conditions (modulo sets of measures 0): (i) $\{2^i\omega : j \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$, (ii) $\{\omega + 2k\pi : k \in \mathbb{Z}\}$ is a disjoint family.

(The proof of this is the same as the one given for Theorem (2.5) in Chapter 7 of [HW]). It follows from the above characterization that if ω is a TFW-set, then any ψ satisfying (4.1) is a TFW, necessarily MSF. We have to recall certain notions from [PSWX]. Given a TFW ψ_0, let

$$\mathcal{M}_{\psi_0}^{TF} = \{\psi \in L^2(\mathbb{R}) : \exists\nu, \text{ such that } \hat{\psi}(\xi) = \nu(\xi)\hat{\psi}_0(\xi)\},$$

where ν is unimodular($|\nu(\xi)| = 1$) and $\nu(2\xi)\overline{\nu(\xi)}$ is 2π-periodic. It has been shown in [PSWX] that sets $\mathcal{M}_{\psi_0}^{TF}$ are pathwise connected. Given an MSF

TFW ψ, let ψ_0 be defined by

$$\hat{\psi}_0(\xi) = |\hat{\psi}(\xi)|.$$

By the above remark, ψ_0 is also an MSF TFW. Furthermore, there exists a unimodular 2π periodic function ν such that

$$\hat{\psi}(\xi) = \nu(\xi)\hat{\psi}_0(\xi),$$

($\nu(\xi)$ is defined by $\dfrac{\hat{\psi}(\xi)}{\hat{\psi}_0(\xi)}$ whenever $\hat{\psi}_0(\xi) \neq 0$, and can be extended as 2π-periodic and unimodular by the property (ii) of TFW-sets). Thus,

$$\psi \in \mathcal{M}^{TF}_{\psi_0}.$$

A look at the proof of the path-connectivity of $\mathcal{M}^{TF}_{\psi_0}$ in [PSWX] shows that the path consists of MSF TFWs. Thus, ψ can be continuously connected with ψ_0 by a path of MSF TFWs. As a consequence, to connect two MSF TFWs with a continuous path of MSF TFWs it is enough to consider TFWs ψ of the particular form

$$\hat{\psi}(\xi) = \chi_\omega(\xi). \tag{4.2}$$

Let us, therefore, consider two TFW-sets ω_1 and ω_2. Adapting a construction from [S] we are going to construct a path of TFW-sets connecting ω_1 with ω_2, which is continuous in the symmetric difference topology, that is the topology given by the metric:

$$d(A, B) = |(A \setminus B) \cup (B \setminus A)|$$

where $|\cdot|$ is the Lebesgue measure. Clearly, this implies that the path of TFWs, associated with TFW-sets by (4.2) is continuous in $L^2(\mathbb{R})$. Let us define two functions:

$$\tau : \mathbb{R} \to (-\pi, \pi] \text{ and } \delta : \mathbb{R} \setminus \{0\} \to (-2\pi, -\pi] \cup [\pi, 2\pi).$$

$\tau(x)$ is defined as the unique number in $(-\pi, \pi]$ which differs from x by an integer multiple of 2π, and $\delta(x)$ is the unique number in $(-2\pi, -\pi] \cup [\pi, 2\pi)$ which is a multiple of x by an integral power of 2. The statement that ω_1 and ω_2 are TFW-sets can be now restated equivalently:

$$\begin{aligned} \tau : \omega_i \to & \quad (-\pi, \pi] \text{ is an injection}, i = 1, 2 \\ \delta : \omega_i \to & \quad (-2\pi, -\pi] \cup [\pi, 2\pi) \text{ is a bijection}, i = 1, 2. \end{aligned}$$

Let

$$\begin{aligned} \tau(\omega_i) &= A_i \subset (-\pi, \pi], \\ \tau_i &= \tau_{|\omega_i}, \\ \delta_i &= \delta_{|\omega_i}, \end{aligned}$$

$i = 1, 2$. Both τ_i and δ_i are homeomorphisms in the symmetric difference topology. The construction of the path follows in steps. Let $t \in [0, \pi]$ be the parameter, and $O_t \subset \omega_1$, $Q_t \subset \omega_2$ be defined by

$$\begin{aligned} O_t &= \tau_1^{-1}(A_1 \cap ((-\pi, -t] \cup (t, \pi])), \\ Q_t &= \tau_2^{-1}(A_2 \cap (-t, t]). \end{aligned}$$

$O_t \cup Q_t$ is a continuous path of sets connecting ω_1 $(t = 0)$ with ω_2 $(t = \pi)$. Since $O_t \cap Q_t = \emptyset$ thus, τ is an injection on $O_t \cup Q_t$. Clearly, $O_t \cup Q_t$ might not be TFW-sets: $\delta_{|O_t \cup Q_t}$ might be neither $1-1$ nor onto. We fix this in two steps. Let $U_t \subset \omega_1$ and $V_t \subset O_t \cup Q_t$ be defined by

$$U_t = \delta_1^{-1}(\delta(O_t) \cap \delta(Q_t)),$$

$$V_t = \tau^{-1}((-min(t, \pi - t), min(t, \pi - t))) \cap (O_t \cup Q_t)$$

and, finally,

$$S_t = (O_t \cup Q_t) \setminus (U_t \cup V_t).$$

We have removed from $O_t \cup Q_t$ the part that violated the injectivity of $\delta_{|O_t \cup Q_t}$, and, additionally, created "some room" by removing V_t. Thus, S_t is a continuous path of sets from ω_1 to ω_2 with the property, that both $\tau_{|S_t}$ and $\delta_{|S_t}$ are injections for $t \in [0, \pi]$. In addition

$$\tau(S_t) \cap (-min(t, \pi - t), min(t, \pi - t)) = \emptyset.$$

Each S_t will now be augmented (in a continuous way so that it becomes a TFW-set). We will use the following lemma.

Lemma 4.3 *([S], Lemma 3.11) Suppose* $A \subset (-2\pi, -\pi] \cup [\pi, 2\pi)$, *and* $s > 0$. *Then, there exists a set* $N(A, s)$, *continuously depending on* A *and* s *(jointly) such that (a)* $N(A, s) \subset (-s, s)$. *(b)* $\delta(N(A, s)) = A$.

Let

$$M_t = N(((-2\pi, -\pi] \cup [\pi, 2\pi)) \setminus \delta(S_t), min(t, \pi - t)).$$

Clearly, $S_t \cup M_t$ is a continuous path of TFW-sets connecting ω_1 with ω_2. We have thus proved

Theorem 4.4 *The set of MSF TFWs is pathwise connected.*

We have obtained only particular results concerning the connectivity of certain subclasses of TFW's. We shall investigate this further in the near future.

References

[BL] Benedetto, J. J. and Li, S., *The Theory of Multiresolution Analysis Frames and Applications to Filter Banks*, Appl. & Comp. Harmonic Analysis, 5, (1998) pp. 389-427.

[BT] Benedetto, J. J. and Treiber, O. M., *Wavelet Frames: Multiresolution Analysis and Extension Principles*, To appear in Wavelet Transforms and Time-frequency Signal Analysis, L. Debnath, editor.

[BRS] Bownik, M., Rzeszotnik, Z. and Speegle, D. *A characterization of Dimension functions of wavelets*, Appl. & Comp. Harmonic Analysis 10, (2001), pp. 71-92.

[D] Daubechies, I., *Ten Lectures on Wavelets*, CBMS-NSF Regional Conf. Series in Appl. Math., Vol. 61, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[HW] Hernández, E. and Weiss, G. *A First Course on Wavelets* , CRC Press, Boca Raton, FL, 1996.

[R] Rzeszotnik, Z. *Characterization Theorems in the Theory of Wavelets*, Ph. D. Thesis, Washington University, 2000.

[S] Speegle, D. M. *The s-elementary wavelets are path-connected*, Proc. Amer. Math. Soc. 127(1999), No. 1, pp. 223-233

[PSW] Papadakis, M., Šikić, H. and Weiss, G. *The Characterization of Low Pass Filters and Some Basic Properties of Wavelets, Scaling Functions* *and Related Concepts*, The Journal of Fourier Analysis and Application 5(1999), pp. 495-521.

[PSWX] Paluszyński,M., Šikić, H., Weiss, G. and Xiao, S. *Generalized Low Pass Filters and MRA Frame Wavelets*, Journal of Geometric Analysis, Vol. 11, No. 2, 2001. pp. 311-342.

[RS] Ron, A. and Shen, Z., *Affine Systems in* $L^2(\mathbb{R}^d)$ *II: Dual Systems.* The Journal of Fourier Analysis and Application 3(1997), pp. 617-637.

[WUTAM] The Wutam Consortium, *Basic Properties of Wavelets*, The Journal of Fourier Analysis and Applications, 4(1998), pp. 575-594.

[WW] Weiss, G. and Wilson, E. N., *The Mathematical Theory of Wavelets*, Proc. of the NATO-ASI meeting "Harmonic Analysis – A Celebration." Kluwer (publisher), to appear in 2001.

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