Tight frame wavelets and the dimension function

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1 Preliminaries

Let $M : \mathbf{R} \longrightarrow [0,1]$ be a measurable, 2π-periodic function which satisfies

$$M(\xi) + M(\xi + \pi) = 1, \tag{1}$$

for a.e. $\xi \in \mathbf{R}$. Obviously, it is very easy to construct all such functions, by having an arbitrary measurable function on $[-\pi/2, \ \pi/2)$, with values in $[0,1]$, and by extending it through (1) and 2π-periodicity to the entire real line.

We define, for every $n \in \mathbf{N}$, a function $\Phi_n : \mathbf{R} \longrightarrow [0,1]$ by

$$\Phi_n(\xi) = \Phi_{n,M}(\xi) \equiv \prod_{j=1}^{n} M\left(\frac{\xi}{2^j}\right) \cdot \chi_{[-2^n\pi, 2^n\pi)}(\xi) \ , \ \xi \in \mathbf{R} \, . \tag{2}$$

Using the standard "peeling off" argument (see [HW] pp.370-372) it is not difficult to prove that

$$\sum_{j \in \mathbf{Z}} \Phi_n(\xi + 2\pi j) = 1 \text{ a.e.} \, , \tag{3}$$

which, by periodization, implies that

$$\| \, \Phi_n \, \|_{L^1(\mathbf{R})} = 2\pi \ . \tag{4}$$

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Furthermore, for every $\xi \in \mathbf{R}$, the sequence $\{\Phi_n(\xi)\}$ is decreasing starting from n large enough, and, therefore, we define a function $\Phi = \Phi_M : \mathbf{R} \longrightarrow [0,1]$, by

$$\Phi(\xi) \equiv \lim_{n\to\infty} \Phi_n(\xi) \ , \ \xi \in \mathbf{R} . \tag{5}$$

By Fatou's lemma $\Phi \in L^1(\mathbf{R})$ and satisfies

$$\| \Phi \|_{L^1(\mathbf{R})} \le 2\pi \ , \tag{6}$$

as well as

$$\sum_{j\in\mathbf{Z}} \Phi(\xi + 2\pi j) \le 1 \text{ a.e.} \tag{7}$$

An easy consequence of (5) is that $\Phi(\xi) = \Phi_n(\xi)\Phi(\xi/2^n)$ which implies (see [PSW], Lemma 1) that the limit $\lim_{n\to\infty} \Phi(\xi/2^n)$ exists and is either 0 or 1. We shall be interested only in such M for which

$$\lim_{n\to\infty} \Phi_M\left(\frac{\xi}{2^n}\right) = 1 \ , \ \text{for a.e. } \xi \in \mathbf{R}. \tag{8}$$

Finally, an easy consequence of (6) and (7) is that

$$\| \Phi \|_{L^1(\mathbf{R})} = 2\pi \Longleftrightarrow \sum_{j\in\mathbf{Z}} \Phi(\xi + 2\pi j) = 1 \text{ a.e.} \tag{9}$$

We shall see how these simple observations valid for nonnegative functions have important consequences in wavelet theory; when combined with the powerful concept of multipliers.

2 Tight frame wavelets

As we know, a function ψ in $L^2(\mathbf{R})$ is an orthonormal wavelet if a system $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbf{Z}$, is an orthonormal basis of $L^2(\mathbf{R})$. Suppose now that we want to drop some requirements from this system; i.e., we do not necessarily want it to be neither orthonormal nor a basis. More precisely, we shall be happy if it keeps the reproducing property: for all $f \in L^2(\mathbf{R})$

$$f = \sum_{j,k\in\mathbf{Z}} \langle f, \, \psi_{jk}\rangle \psi_{jk} \tag{10}$$

unconditionally in $L^2(\mathbf{R})$. This happens to be equivalent to the condition

$$\| f \|_2^2 = \sum_{j,k\in\mathbf{Z}} |\langle f, \, \psi_{jk}\rangle|^2 \tag{11}$$

for every $f \in L^2(\mathbf{R})$, i.e., to the requirement that $\{\psi_{jk}\}$ is a *normalized tight frame* for $L^2(\mathbf{R})$ (see Chapters 7 and 8 in [HW] for details). For short, following [PSWX1], we shall say that ψ is a TFW. Clearly, every orthonormal wavelet is a TFW. The converse, however, is not true (see [PSWX1] and [PSWX2] for various examples). Actually, a TFW can satisfy various levels of orthogonality. For example, ψ is a *semiorthogonal TFW* if its various "resolution" levels are orthogonal, i.e., if $\psi_{j_1k_1} \perp \psi_{j_2k_2}$, whenever $j_1 \neq j_2$.

The technique of wavelet multipliers extends nicely to TFW-s. Following [PSWX1], we shall say that ν is a *TFW-multiplier* if $(\nu\hat{\psi})\check{}$ is a TFW, whenever ψ is a TFW; $\hat{f}$ stands for the Fourier transform given in the form

$$\hat{f} = \int_{\mathbf{R}} f(x) \mathrm{e}^{-i\xi x}\, dx, \;\; f \in L^1(\mathbf{R}).$$

One of the key results in [PSWX1] is that the set of TFW-multipliers equals that of (orthonormal) wavelet-multipliers (defined in an analogous way; see [WUTAM]), and consists of unimodular functions ν such that $\nu(2\xi)\overline{\nu(\xi)}$ is 2π-periodic.

The crucial feature of TFW-s is that they allow for a very general extension of the notion of an MRA wavelet (see [HW] for definitions in the case of orthonormal wavelets). We need some definitions first (see [PSWX1] for details). A 2π-periodic, measurable function $m : \mathbf{R} \longrightarrow \mathbf{C}$ is a *generalized filter* if the corresponding function $M(\xi) \equiv |m(\xi)|^2$ satisfies (1). A function $\varphi \in L^2(\mathbf{R})$ is a *pseudo-scaling function* if there exists a generalized filter m (not necessarily unique) such that

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi), \tag{12}$$

for a.e. $\xi \in \mathbf{R}$. A TFW ψ is an MRA TFW if there exist a pseudo-scaling function φ and a corresponding generalized filter m such that

$$\hat{\psi}(2\xi) = \mathrm{e}^{i\xi}\overline{m(\xi+\pi)}\hat{\varphi}(\xi), \tag{13}$$

for a.e. $\xi \in \mathbf{R}$. As it turns out, the generalized filter in (13) is forced to be even more - *a generalized low pass filter*, i.e., a generalized filter m such that the corresponding M satisfies (8). Notice also that a consequence of (12) is that

$$\Phi(\xi) = |\,\hat{\varphi}(\xi)\,|^2, \tag{14}$$

for a.e. $\xi \in \mathbf{R}$.

The multiplier approach enables us to go "in reverse". More precisely, consider an arbitrary generalized low pass filter m. Then, there is a unimodular, 2π-periodic function μ (*a filter multiplier*) such that $m = \mu|m|$. Using (14) and (5) we then define the corresponding $|\hat{\varphi}|$, and after that we define (by solving $\nu(2\xi)\overline{\nu(\xi)} = \mu(\xi)$ for ν) $\hat{\varphi}$ by

$$\hat{\varphi}(\xi) \equiv \nu(\xi) \cdot \sqrt{\Phi(\xi)} \ . \tag{15}$$

Then we use (13) to obtain an MRA TFW ψ. Therefore, from an arbitrary generalized low pass filter we can construct an MRA TFW whose corresponding filter is exactly the one we started with. The construction shows that we can restrict our attention to nonnegative functions, where the procedure is much simpler (see also our first section).

3 Dimension function

By extending the notion of the dimension function from the case of orthonormal wavelets (see section 7.3. in [HW]) we define (following [PSWX2]) the *dimension function* D_ψ of a TFW ψ by

$$D_\psi \equiv \sum_{j=1}^{\infty} \sum_{k\in\mathbf{Z}} \left| \hat{\psi}(2^j(\xi + 2k\pi)) \right|^2 , \tag{16}$$

for $\xi \in \mathbf{R}$. Clearly, D_ψ is a 2π-periodic function which is finite a.e., since

$$\int_{-\pi}^{\pi} D_\psi(\xi)\, d\xi = \| \hat{\psi} \|_2^2 \le 2\pi \ . \tag{17}$$

On this level of generality D_ψ is not necessarily integer-valued (see [PSWX2] for detailed study), but there is a closely related integer-valued function. Consider vectors $\Psi_j(\xi) \in \ell^2(\mathbf{Z})$, $j \ge 1$, given by

$$\Psi_j(\xi) \equiv \Big(\hat{\psi}(2^j(\xi + 2k\pi)) \ : \ k \in \mathbf{Z} \Big) \ ; \ \xi \in \mathbf{R}, \tag{18}$$

and the function $\dim_\psi(\xi)$ given by

$$\dim_\psi(\xi) \equiv \dim \overline{\operatorname{span}\{ \Psi_j(\xi) \ : \ j \ge 1 \}} \, , \ \xi \in \mathbf{R}. \tag{19}$$

The function $\dim_\psi$ is 2π-periodic, but it is integer-valued. Using linear algebra arguments it is not difficult to prove (see [RS] for even more general statement) that for a TFW ψ

$$D_\psi(\xi) \le \dim_\psi(\xi), \tag{20}$$

for a.e. $\xi \in \mathbf{R}$. The following theorem is proved in [PSWX2].

Theorem 3.1 *Suppose ψ is a TFW. The following are equivalent:*

a) ψ is a semiorthogonal TFW;

b) $D_\psi = \dim_\psi$ a.e.;

c) D_ψ is integer-valued.

Finally, let us turn our attention to the MRA case.

4 MRA TFW-s

Unlike the orthonormal case, in the case of TFW-s we can not decide easily, in terms of D_ψ, when is ψ an MRA. However, we know (Theorem 3.8 in [PSWX2]) that for a TFW ψ the following is valid:

$$\psi \text{ is an MRA TFW} \iff \dim_\psi(\xi) = 0 \text{ or } 1, \text{ for a.e. } \xi \in \mathbf{R}. \tag{21}$$

Let us, however, turn our attention to filters. Suppose m is a generalized low pass filter. From a *probabilistic* point of view m generates for each $\xi \in \mathbf{R}$ a sequence of probability measures $\{ P_\xi^n \}$ on $(\mathbf{Z}, \mathcal{P}(\mathbf{Z}))$ through

$$P_\xi^n(\{k\}) = \Phi_n(\xi + 2\pi k) \ , \ k \in \mathbf{Z}; \tag{22}$$

as provided by (3) - indeed, the "peeling off" argument can be considered as the standard Markov chain argument (see [DGH] and [G] for many features of such an approach). However, in the limiting case we obtain only subprobability measures P_ξ (through $\Phi(\xi)$) - see (7). Observe also that for any MRA TFW ψ, obtained from m through (13), we have

$$D_\psi(\xi) = P_\xi(\mathbf{Z}), \tag{23}$$

for a.e. $\xi \in \mathbf{R}$. We get now, as an easy consequence of section 1 results, the following theorem; which contains several results obtained in [PSW], [DGH], [PSWX2].

Theorem 4.1 *Suppose m is a generalized low pass filter. Then the following are equivalent:*

a) m is a low pass filter of an (MRA) orthonormal wavelet;

b) $P_\xi(\mathbf{Z}) = 1$, for a.e. $\xi \in \mathbf{R}$;

c) $\| \Phi \|_{L^1(\mathbf{R})} = 2\pi$;

d) $D_\psi(\xi) = 1$, for a.e. $\xi \in \mathbf{R}$.

Remark 24 *The condition* $\dim_\psi(\xi) = 1$ *a.e. is not equivalent to the four conditions given above.*

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