The algebra of shift invariant spaces and applications to wavelet

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Shift Invariant Spaces and their Dyadic Dilates*

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Abstract: The study of wavelet and more general reproducing function systems revolves around the properties of shift invariant spaces. While a great deal is known about such spaces, there are also many open questions. In particular, every classification theorem for wavelet systems is also a classification theorem for a certain type of shift invariant space–to date, few such classification theorems are known. Both for theoretical and computational efforts, it is convenient to regard shift invariant spaces as analogs of inner product spaces with periodic functions as "scalars" and with a certain function-valued form replacing the inner product. We will discuss a variety of results on shift invariant spaces and indicate some of the open research questions.

1 DEFINITIONS AND MOTIVATION

1.1 Shift Invariant Space Definitions

For $n \in \mathbf{N}$ and $k \in \mathbf{Z}^n$, $T_k\psi(x) = \psi(x-k)$ defines the translation (or shift) operator T_k on $L^2(\mathbf{R}^n)$.

V is a *shift invariant space* (SIS) if V is a closed subspace of $L^2(\mathbf{R}^n)$ and $T_k(V) \subset V$ for every $k \in \mathbf{Z}^n$. The shift invariant space $< \varphi_1, \varphi_2, \dots, \varphi_N >$ *generated* by $\varphi_1, \varphi_2, \dots, \varphi_N$ is the smallest SIS containing each φ_i and is thus the closure in $L^2(\mathbf{R}^n)$ of the span of $\{T_k\varphi_i \mid 1 \le i \le N, k \in \mathbf{Z}\}$.

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1.2 TFW Definition

For $A \in GL(n, \mathbf{R})$, the *dilation operator* D_A is the unitary operator on $L^2(\mathbf{R}^n)$ defined by $D_A f(x) =| detA |^{1/2} f(Ax)$. Then, for $L \in \mathbf{N}$,

$$\Psi = \{\psi_{j,k}^{(l)} = D_A^j T_k \psi^{(l)} \mid 1 \leq l \leq L, j \in \mathbf{Z}, k \in \mathbf{Z}^n\}$$

is a *tight frame wavelet system* (TFW) with L generators and dilation matrix A if

$$|| f ||^2 = \sum_{l=1}^{L} \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}^n} |< f, \psi_{j,k}^{(l)} >|^2$$

for all $f \in L^2(\mathbf{R}^n)$. We can then describe each f weakly in the form

$$f = \sum_{j,k,l} < f, \psi_{j,k}^{(l)} > \psi_{j,k}^{(l)}.$$

A *dyadic TFW* involves dilation by powers of two, i.e. $A = 2I_n$.

1.3 Remarks

In the context of Definition 1.2, the building block for the TFW system is the shift invariant space $V =< \psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(n)} >$. We then have $L^2(\mathbf{R}^n)$ spanned by the images of V under the integer powers of D_A. As a starting point for construction/classification of TFWs, we need to know how to construct/classify all shift invariant spaces with this spanning property.

Gabor systems and hybrid reproducing function systems unifying TFW and Gabor systems also involve a finitely generated SIS V as a building block but replace the family of integer powers of a fixed dilation operator with a countable family of operators each of which is a product of a dilation operator and a modulation operator.

The theory of shift invariant spaces goes back to the work of Helson roughly 40 years ago. Contributions to the theory have been made by virtually every researcher working on wavelet and Gabor systems. Many of the properties described in Section 2 were obtained with very different methods by Ron and Shen. The papers of de Boor, De Vore, Ron;Lawrence Baggett; Marcin Bownik; and Ziemowit Rzeszotnik should also be mentioned. The very algebraic approach described in Section 2 differs from the methods used by these authors.

2 BASIC PROPERTIES OF SHIFT INVARIANT SYSTEMS

2.1 Definitions

We will take

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx$$

as the definition of the Fourier transform operator $\mathcal{F}f = \hat{f}$ with $\check{g} = \mathcal{F}^{-1}g$. Functions on $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ are identified with $\mathbf{Z}^n$-periodic functions on $\mathbf{R}^n$ and subsets of $\mathbf{T}^n$ are identified with subsets of $\mathbf{R}^n$ invariant under $\mathbf{Z}^n$ translations. Haar measure on $\mathbf{T}^n$ may be regarded as the restriction to the unit cube of Lebesgue measure $d\xi$ on $\mathbf{R}^n$.

1. The frequency orbit support Ω_f of $f \in L^2(\mathbf{R}^n)$ is the set of all $\xi \in \mathbf{R}^n$ for which there exists $k \in \mathbf{Z}^n$ such that $\hat{f}(\xi + k) \neq 0$. Up to a set of measure 0, Ω_f is a well defined subset of $\mathbf{T}^n$.

2. For $m \in L^\infty(\mathbf{R}^n)$, the *module action* of m on f is defined by $m \cdot f = g$ where $\hat{g} = m\hat{f}$ is the pointwise product of m and $\hat{f}$.

3. For $f, g \in L^2(\mathbf{R}^n)$, we define the *bracket product* by

$$[\hat{f}, \hat{g}](\xi) = \sum_{k \in \mathbf{Z}^n} \hat{f}(\xi + k)\overline{\hat{g}}(\xi + k).$$

2.2 Lemma

The bracket operator $(f, g) \mapsto [\hat{f}, \hat{g}]$ is bounded as a map from $L^2(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ into $L^1(\mathbf{T}^n)$ and is hermitian, $L^\infty(\mathbf{R}^n)$-sesquilinear, and positive semi-definite. Specifically,

1. For all $f, g \in L^2(\mathbf{R}^n)$, $[\hat{f}, \hat{g}] = \overline{[\hat{g}, \hat{f}]}$ is in $L^1(\mathbf{T}^n)$ with

$$|| [\hat{f}, \hat{g}] ||_{L^1(\mathbf{T}^n)} \leq || f ||_{L^2(\mathbf{R}^n)} || g ||_{L^2(\mathbf{R}^n)}$$

2. $(f, g) \mapsto [\hat{f}, \hat{g}]$ is $\mathbf{R}$-bilinear with $[m\hat{f}, \hat{g}] = m[\hat{f}, \hat{g}] = [\hat{f}, \overline{m}\hat{g}]$ for all f, g and all $m \in L^\infty(\mathbf{T}^n)$.

3. $[\hat{f}, \hat{f}] \geq 0$ a.e with $[\hat{f}, \hat{f}](\xi) > 0$ for $\xi \in \Omega_f$.

Proof. Elementary verifications.

2.3 Lemma

For $f, g \in L^2(\mathbf{R}^n)$ and $k \in \mathbf{Z}^n$, $< T_k f, g >$ is the k-th Fourier coefficient $\int_{\mathbf{T}^n} [\hat{f}, \hat{g}](\xi) e^{-2\pi i \xi \cdot k} d\xi$ of the periodic function $[\hat{f}, \hat{g}]$.

Proof. With $e_k(\xi) = e^{-2\pi i \xi \cdot k}$, the definition of the Fourier transform along with unitarity of $\mathcal{F}$ yields $< T_k f, g > = < e_{-k} \hat{f}, \hat{g} > = < [\hat{f}, \hat{g}], e_k >_{L^2(\mathbf{T}^n)}$.

2.4 Proposition

Let V be a closed subspace of $L^2(\mathbf{R}^n)$. Then

1. V is a SIS $\Leftrightarrow$ V is an $L^\infty(\mathbf{T}^n)$-module.
2. If $V = < \varphi >$ is generated by $\varphi \neq 0$, then the orthogonal projection from $L^2(R^n)$ onto V is given by $f \mapsto [\hat{f}, \hat{\psi}]\psi$ where ψ is the function for which $\hat{\psi}$ vanishes outside Ω_φ and

$$\hat{\psi}(\xi) = \frac{\hat{\varphi}(\xi)}{[\hat{\varphi}, \hat{\varphi}]^{1/2}(\xi)}$$

for $\xi \in \Omega_\varphi$. Moreover, $g \in V \Leftrightarrow g = m \cdot \psi$ for some $m \in L^2(\mathbf{T}^n)$.

Proof. Suppose V is a SIS. Since the operators T_k are unitary, the orthogonal complement $V^\perp$ of V is also a SIS. By Lemma 2.3, $V^\perp$ consists of all $g \in L^2(\mathbf{R}^n)$ for which $[\hat{f}, \hat{g}] = 0 \, \forall f \in V$. It follows from Lemma 2.2(2) that V is a $L^\infty(\mathbf{T}^n)$-module. The converse is trivial since $T_k f = e_{-k} \cdot f$. This proves (1) and (2) follows easily from Lemma 2.2.

2.5 Consequences

- In dealing with shift invariant spaces, the module language and bracket operation allow us to put aside the awkwardness of calculations with countably many translation operators applied to a fixed function φ. Instead, we can treat singly generated shift invariant spaces $< \varphi >$ as being "one-dimensional" over the ring $L^\infty(\mathbf{T}^n)$ and can think of the bracket operator as being a ring-valued "inner product". In particular, replacement of φ with ψ in Proposition 2.4 is the analog of dividing a non-zero vector by its length to obtain a unit vector in any inner product space. Also the $< \varphi >$-component $m \cdot \psi$ of any function f is given by $m = [\hat{f}, \hat{\psi}]$ in analogy with the usual orthogonal projection formula in an inner product space. Although, in view of Lemma

2.3, there is a Fourier series interpretation of every formula involving brackets, the Lemma 2.2 properties are all we need to do calculations with shift invariant spaces and we therefore need not interrupt such calculations with Fourier series arguments.

- All of the usual elementary constructions in inner product spaces can be imitated in shift invariant spaces. Since the Hilbert space $L^2(\mathbf{R}^n)$ is separable, every closed subspace is countably generated over the complex numbers and hence every shift invariant space V is countably generated over the ring $L^\infty(\mathbf{T}^n)$. Given a set $\varphi_1, \varphi_2, \ldots$ of generators for V, we may as well assume that $\varphi_{i+1} \notin < \varphi_1, \varphi_2, \ldots, \varphi_i > \forall i$. Then merely copying down the usual formulas for the Gram-Schmidt process and replacing inner products with brackets gives us the orthogonal direct sum decomposition $V = \bigoplus_{i\geq 1} < \psi_i >$ with $\psi_i \in < \varphi_1, \varphi_2, \ldots, \varphi_i > \forall i$ and $f \mapsto \sum_{i\geq 1}[\hat{f}, \hat{\psi}_i] \cdot \psi_i$ is the orthogonal projection map from $L^2(\mathbf{R}^n)$ onto V. We will call any system $\psi_1, \psi_2, \ldots$ with this property a *system of orthogonal generators* (SONG) for V.

- The sole need for caution with proceeding with additional analogs of inner product spaces is the fact that while all unit vectors have length 1, our "unit shift functions" ψ have $[\hat{\psi}, \hat{\psi}] = \chi_{\Omega_\psi}$. This means that two SONGS for V need not have the same number of elements. Indeed, whenever Ω_φ is the disjoint union of sets $\Omega_i, i \geq 1$, we have $< \varphi >= \bigoplus_{i\geq 1} < \chi_{\Omega_i} \cdot \varphi >$. Establishing some control over the support sets of a SONG is therefore the crucial difference between shift invariant spaces and inner product spaces. To do this, we first need a reasonable notion of "dimension" for shift invariant spaces.

2.6 Corollary

Let V be any SIS. Then there is a measurable function $\dim_{\mathrm{V}}$ from $\mathbf{T}^n$ into $\mathbf{N} \cup \{0, \infty\}$ such that for every SONG $\phi_i, 1 \leq i \leq N$, for V, $\dim_{\mathrm{V}} = \sum_{\mathrm{i}}^{\mathrm{N}} \chi_{\Omega_{\mathrm{i}}}$ almost everywhere.

Proof. Suppose $\varphi_i, 1 \leq i \leq N$, and $\psi_j, 1 \leq j \leq M$, are two SONGs for V. Using Proposition 2.4 with $=$ understood to mean $=$ *a.e.*,

$$\sum_{i=1}^{N} \chi_{\Omega_{\varphi_i}} = \sum_{i=1}^{N} [\hat{\varphi}_i, \hat{\varphi}_i] = \sum_{j=1}^{M}\sum_{i=1}^{N} \mid [\hat{\varphi}_i, \hat{\psi}_j] \mid^2 = \sum_{j=1}^{M} [\hat{\psi}_j, \hat{\psi}_j] = \sum_{j=1}^{M} \chi_{\Omega_{\psi_j}}.$$

Corollary 2.6 follows.

2.7 Definitions

For V a SIS, the *fundamental support sets* $\Omega_1, \Omega_2, \ldots$ of V are defined up to sets of measure zero by $\Omega_i = \{\xi \mid \dim_V \geq i\}$. The *rank* or *dimension* of V is the cardinal number $d_V = \| \dim_V \|_\infty$. [Some authors call $\dim_V$ the multiplicity function on V and then call d_V the multiplicity of V]

2.8 Theorem

(Adapted from deBoor, De Vore, Ron) For V a SIS, d_V is the minimal cardinality for generating sets for V. If $\varphi_i, i \geq 1$, is any SONG for V for which $\Omega_{\varphi_i} \supset \Omega_{\varphi_{i+1}} \forall i$, then each Ω_{φ_i} is the i^{th} fundamental support set for V. Conversely, there is a "rearrangement" algorithm converting any SONG for V to a SONG with this ordering property for its support sets.

2.9 Primary Decomposition Theorem

Let V be any SIS and $\mathcal{D}$ the set of positive values assumed by $\dim_V$ with positive measure. Thus, with Ψ_j the set of all $\xi \in \mathbf{T}^n$ for which $\dim_V(\xi) = j$, the first fundamental support set for V is the disjoint union of the sets $\Psi_j, j \in \mathcal{D}$. Then V has the canonical decomposition $V = \bigoplus_{j \in \mathcal{D}} V_j$ where $\forall j \in \mathcal{D}$, j is the rank of the shift invariant subspace V_j and each of the j fundamental support sets for V_j coincides, modulo a null set, with Ψ_j. In algebraic terminology, the submodules V_j are free of dimension j over the ring $L^\infty(\Psi_j)$ of essentially bounded periodic functions vanishing outside Ψ_j.

2.10 Remarks

1. The proofs of Theorems 2.8 and 2.9 are somewhat lengthy but not intrinsically difficult since they rely exclusively on the easy properties we have established in 2.2-2.6.

2. There is a close analog between Theorems 2.8-2.9 and the familiar cyclic and generalized eigenspace decomposition theorems for an operator on a finite dimensional vector space. The analogy arises by contrasting principal ideals generated by characteristic functions in the ring $L^\infty(\mathbf{T}^n)$ with ideals in the ring of polynomials in one variable. In essence, the same kinds of algebraic arguments which underline the theory of modules over a principal ideal domain carry over with only small changes to shift invariant spaces even though $L^\infty(\mathbf{T}^n)$ is not a principal ideal domain.

3. The shift invariant space $L^2(\mathbf{R}^n)$ is free of rank ∞ over $L^\infty(\mathbf{T}^n$. The simplest choice of a SONG with the support ordering property consists of the inverse Fourier transforms of the characteristic functions of unit cubes with vertices in $\mathbf{Z}^n$. One can then easily construct shift invariant spaces satisfying Theorem 2.9 for any preassigned set $\mathcal{D}$ of positive integers and any choice of disjoint subsets $\Psi_j, j \in \mathcal{D}$ of $\mathbf{T}^n$–indeed, all one has to do is to select for each $j \in \mathcal{D}$ a collection of j unit cubes with vertices in $\mathbf{Z}^n$, intersect each such cube with Ψ_j, and take as a SONG the inverse Fourier transforms of the associated characteristic functions. At least in principal, one can proceed to describe via change of basis matrices all of the shift invariant spaces sharing the same set of dimension values and the same support sets. We will leave the details of this to the reader.

3 DYADIC SHIFT INVARIANT SPACES

3.1 Notation

Throughout this section we will denote by D the dyadic dilation operator $Df(x) = 2^{n/2}f(2x)$ on $L^2(\mathbf{R}^n)$.

3.2 Definition

A shift invariant space is *dyadic* if $V \subset D(V)$ or, equivalently, $D^{-1}V \subset V$.

3.3 Remarks

1. If V is a SIS, then $D(V)$ is a SIS in view of the commutation identity $T_kD = DT_{2k}$ valid for all k.
2. For any $\psi \in L^2(\mathbf{R}^n)$, the closure V_+ of the span of $\{D^jT_k\psi = \psi_{j,k} \mid j \in \mathbf{Z}, k \in \mathbf{Z}^n\}$ is a SIS with $DV_+ \subset V_+$. Therefore the orthogonal complement V_- of V_+ is a dyadic SIS.
3. For ψ as above, the SIS V_ψ generated by $\{\psi_{j,0} \mid j \leq 0\}$ is dyadic with
$$V_\psi + < \psi > \subset D(V_\psi).$$
When $\{\psi_{j,k}\}$ is a dyadic TFW which is *semi-orthogonal* in the sense that $\psi_{j,k} \perp \psi_{j',k'}$ whenever $j \neq j'$, it follow that the closure of the subspace spanned by $\{\psi_{j,k} \mid j < 0, k \in \mathbf{Z}^n\}$ coincides with both V_ψ and V_-.

3.4 Basic Questions Concerning Dyadic TFWs

- Question 1. Can we use the general theory of shift invariant spaces to construct all finitely generated dyadic SISs? Which of these spaces has the additional property that

$$L^2(\mathbf{R}^n) = \overline{\sum_{j\in\mathbf{Z}} D^j V}?$$

- Question 2. Which dyadic SISs admit a function ψ for which $D(V) = V \oplus < \psi >$? What else must be true in order that ψ generates a dyadic TFW with $V{=}V_\psi$?

- Question 3. Given any finitely generated SIS $V =< \varphi_1, \varphi_2, \ldots, \varphi_N >$ what is the relationship between the rank and fundamental support sets for V and $D(V)$?

- Question 4. Is there a concrete way to "renormalize" any dyadic SIS into one which is semi-orthogonal?

A team consisting of Eugenio Hernández, Hrvoje Šikić, Fernando Soria, Guido Weiss, and the author have made considerable progress in answering Questions 1-4 and related questions in the case $n = 1$. This gives reason to hope that a reasonably complete theory of dyadic TFWs may be within reach. At least partial generalizations to arbitrary n for expanding integral dilation matrices can be anticipated.

3.5 Theorem(W)

For $\Omega \subset \mathbf{T}^n$, there exists a measurable subset $S \subset \mathbf{R}^n$ for which $\Omega = \{\xi + k \mid \xi \in S, k \in \mathbf{Z}^n\}$ and $\frac{1}{2}S \subset S \Leftrightarrow$ there exists φ for which $< \varphi >$ is a dyadic SIS with $\Omega_\varphi = \Omega$.

Proof. The proof of the implication $\Leftarrow$ is very easy. Suppose $< \varphi >$ is a dyadic SIS. It follows from $D^{-1}\varphi \in < \varphi >$ that there exists a periodic function m for which $\hat{\varphi}(\xi) = m(\xi/2)\hat{\varphi}(\xi/2)$ a.e. Then the set S of all ξ for which $\hat{\varphi}(\xi) \neq 0$ satisfies $\frac{1}{2}S \subset S$ and $\Omega_\phi = \{\xi + k \mid \xi \in S, k \in \mathbf{Z}^n\}$. The proof of the converse involves a series of technical adjustments to convert a given set S with the indicated properties to a new set S' with the same properties and on which we can build an appropriate function $\hat{\varphi}$.

3.6 Theorem(W)

Suppose $n = 1$ and $V = < \varphi > \subset L^2(\mathbf{R})$ with $\Omega = \Omega_\varphi$. Define

$$\Omega_1 = 2\Omega \cup (2\Omega + 1),$$

$$\Omega_2 = 2\Omega \cap (2\Omega + 1),$$

$$\hat{\varphi_1} = \frac{\widehat{D\varphi}}{[\widehat{D\varphi}, \widehat{D\varphi}]^{1/2}}, \text{and}$$

$$\hat{\varphi_2} = \chi_{\Omega_2} e_{1/2} \widehat{D\varphi}.$$

Then

1. $D(V)) = < D(\varphi) > \Leftrightarrow \Omega_2$ is a null set.
2. When Ω_2 has positive measure, Ω_1 and Ω_2 are the fundamental support sets for $D(V)$ and $\{\varphi_1, \varphi_2\}$ is a SONG for $D(V)$.

The proof is a reasonably straightforward calculation. There is a somewhat cumbersome generalization to $L^2(\mathbf{R}^n)$.

3.7 Corollary

For any finitely generated SIS $V \subset L^2(\mathbf{R}^n)$, $d_V \leq d_{D(V)} \leq 2d_V$ and $d_{D(V)}$ is completely determined by intersection relations among the fundamental support sets for V and their translations by $1/2$. This leads to recipes for constructing examples of dyadic SISs with $d_{D(V)} = d_V + 1$.