Hopf Galois Theory

Universitat de Barcelona

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Motivation: In its early form Galois theory studies the action of a group G (the Galois group) on the set of roots X of a given polynomial in one variable, which renders the map

$$G \times X \to X \times X, \quad (g, x) \mapsto (gx, x) \tag{1}$$

bijective, i.e. X is a principal homogeneous space for G.
If $K \subseteq L$ is a finite Galois extension of fields with Galois group G, then the group algebra $K[G]$ acts on L, which induces a coaction $L \to L \otimes_K K^{(G)}$ of the ring of functions $K^{(G)}$ on L. By left L-linear extension we obtain an isomorphism $L \otimes_K L \to L \otimes_K K^{(G)}$. This isomorphism is the motivation for the Hopf Galois theory, where one replaces, roughly speaking, the action of the group G by a coaction $L \to L \otimes_K H$ of a Hopf algebra H on a K-algebra L such that the left L-linear extension

$$L \otimes_K L \to L \otimes_K H \tag{2}$$

is an isomorphism. If L and H are commutative, this means that $\operatorname{Spec} L$ is a principal homogeneous space for the affine group scheme $\operatorname{Spec} H$. In the Hopf Galois theory L is no longer required to be a field, but merely a K-algebra and L and H do not even need to be commutative. Therefore Hopf Galois extensions can be interpreted as principal homogeneous spaces in non-commutative geometry.
In contrast to the classical Galois theory of field extensions, the Hopf algebra H is not anymore uniquely determined by the extension $L|K$ and the Galois correspondence becomes more interesting.

Topics planned to be discussed:

- A short recall of the history of classical Galois theory up to Dedekind and Artin
- Galois theory of commutative rings (with a group acting)
- Introduction to Hopf algebras and comodules

- Hopf Galois extensions of commutative rings
- Descent
- Hopf (bi)Galois extensions of (possibly non-commutative) rings
- Quantum torsors
- Galois correspondence for Hopf (bi)Galois extensions
- Tensor categories
- Representation theory for Hopf algebras and Tannaka duality

Schedule: First class: December 2, 11:00 – 13:00, aula B2, Faculty of Mathematics.
In the first class we will determine the schedule of the following classes. The course is expected to finish by the end of January.

Prerequisite: Basic knowledge of algebra, commutative algebra and category theory. A basic understanding of algebraic geometry is helpful but not necessary.

Instructor: Florian Heiderich (`florian@heiderich.org`)

Literature

[1] Cyril Grunspan. Quantum torsors. *J. Pure Appl. Algebra*, 184(2-3):229–255, 2003.

[2] André Joyal and Ross Street. An introduction to Tannaka duality and quantum groups. In *Category theory (Como, 1990)*, volume 1488 of *Lecture Notes in Math.*, pages 413–492. Springer, Berlin, 1991.

[3] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, RI, 1993.

[4] Peter Schauenburg. Hopf-Galois and bi-Galois extensions. In *Galois theory, Hopf algebras, and semiabelian categories*, volume 43 of *Fields Inst. Commun.*, pages 469–515. Amer. Math. Soc., Providence, RI, 2004.